University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs
Professor: Alistair Savage

Final Exam
December 2017

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	Total
Maximum	4	4	4	4	3	4	5	7	4	4	43
Grade											

QUESTION 1. **[4 points]** For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. (You will not lose points for incorrect answers.)

____ $\forall x, y, z \in \mathbb{Z},\ (zx = zy \implies x = y)$

____ $\forall z \in \mathbb{R}_{\geq 0}\ \big((\forall n \in \mathbb{N},\ z < \frac{1}{n}) \implies z = 0\big)$

____ $\forall x \in \mathbb{R}_{>0},\ \exists y \in \mathbb{R}_{>0}$ such that $y < x$.

____ $\forall n \in \mathbb{N},\ \exists m \in \mathbb{N}$ such that $m < n$.

____ If $A \neq \varnothing$, $A \subseteq B$, and B is bounded above, then $\sup A \leq \sup B$.

____ The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ is injective.

____ The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ is surjective.

____ It is possible for a sequence $(x_n)_{n=1}^{\infty}$ to have limits L and L' with $L \neq L'$.

QUESTION 2. **[4 points]** Write the negations of the following statements. Simplify your answers as much as possible.

(a) For every integer n, there exists an integer m such that m divides n.

(b) $\forall n \in \mathbb{N},\ \exists x \in \mathbb{R}$ such that $x^9 + x = n$.

(c) $\forall a \in \mathbb{Z},\ \big((a^2 \equiv 0 \mod 4) \text{ or } (a^2 \equiv 1 \mod 4)\big)$.

(d) $\exists c \in \mathbb{R}_{>0}$ such that $\forall n \in \mathbb{N}\ (c^2 > n \implies c > n)$.

Question 3. [**4 points**] Consider the following statement:

"For all sets A, B, and C, we have $A - (B \cup C) = (A - B) \cap (A - C)$."

Prove this statement is true or give a counterexample.

QUESTION 4. **[4 points]** Consider the sequence $(x_n)_{n=1}^{\infty}$ defined recursively by

$$x_1 = 18,$$
$$x_{n+1} = x_n^2 + 6, \quad n \in \mathbb{N}.$$

Prove that x_n is divisible by 3 for all $n \in \mathbb{N}$.

QUESTION 5. [**3 points**] Let

$$A = \{2a : a \in \mathbb{Z}\} \qquad \text{and} \qquad B = \{4b + 3 : b \in \mathbb{Z}\}.$$

Prove that $A \cap B = \varnothing$.

QUESTION 6. [**4 points**] Let $\sim$ be the relation on $\mathbb{R} \times \mathbb{R}$ defined by

$$(x, y) \sim (w, z) \iff x + z = y + w.$$

Prove that $\sim$ is an equivalence relation.

QUESTION 7. [**5 points**]

(a) Suppose $a \in \mathbb{N}$ satisfies $a \equiv 1 \mod 25$. Prove that $a \equiv 1 \mod 5$.

(b) Is the statement

$$\forall a \in \mathbb{N}\,(a \equiv 1 \mod 5 \implies a \equiv 1 \mod 25)$$

true or false? Justify your answer.

(c) Find an integer n such that

$$70002 \leq n \leq 70009 \quad \text{and} \quad n^2 \equiv 1 \mod 7.$$

(d) *Bonus*: Find a second integer n with the same properties.

QUESTION 8. [**7 points**] Consider the set

$$A = \left\{ 3 - \frac{6}{x} : x \in \mathbb{R},\ x \geq 1 \right\}.$$

(a) Find the supremum of A or prove that it does not exist.

(b) Find the largest element (i.e. maximum) of A or prove that it does not exist.

(c) Find the smallest element (i.e. minimum) of A or prove that it does not exist.

(d) Find the infimum of A or prove that it does not exist.

QUESTION 9. [**4 points**] Consider the function

$$f\colon \mathbb{R}_{>0} \to \mathbb{R}, \quad f(x) = 2 - \frac{8}{3x}.$$

(a) Is f injective? Justify your answer.

(b) What is the image of f? Write your answer as an interval. You do *not* need to justify your answer.

(c) Is f surjective?

QUESTION 10. [**4 points**]

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.

(b) What is

$$\lim_{n \to \infty} \frac{1}{2n+3} \ ?$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences.

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