University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

Professor: Alistair Savage

Final Exam – Solutions

December 2017

Surname ____________________ First Name ____________________

Student # ____________________ Seat # ____________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	Total
Maximum	4	4	4	4	3	4	5	7	4	4	43
Grade											

QUESTION 1. **[4 points]** For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. (You will not lose points for incorrect answers.)

- F $\forall x, y, z \in \mathbb{Z},\ (zx = zy \implies x = y)$
- T $\forall z \in \mathbb{R}_{\geq 0}\ \left((\forall n \in \mathbb{N},\ z < \frac{1}{n}) \implies z = 0\right)$
- T $\forall x \in \mathbb{R}_{>0},\ \exists y \in \mathbb{R}_{>0}$ such that $y < x$.
- F $\forall n \in \mathbb{N},\ \exists m \in \mathbb{N}$ such that $m < n$.
- T If $A \neq \varnothing$, $A \subseteq B$, and B is bounded above, then $\sup A \leq \sup B$.
- F The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ is injective.
- F The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ is surjective.
- F It is possible for a sequence $(x_n)_{n=1}^{\infty}$ to have limits L and L' with $L \neq L'$.

QUESTION 2. **[4 points]** Write the negations of the following statements. Simplify your answers as much as possible.

(a) For every integer n, there exists an integer m such that m divides n.

Solution: There exists an integer n such that for all integers m, m does not divide n.

(b) $\forall n \in \mathbb{N},\ \exists x \in \mathbb{R}$ such that $x^9 + x = n$.

Solution: $\exists n \in \mathbb{N}$ such that $\forall x \in \mathbb{R},\ x^9 + x \neq n$.

(c) $\forall a \in \mathbb{Z},\ \left((a^2 \equiv 0 \mod 4) \text{ or } (a^2 \equiv 1 \mod 4)\right)$.

Solution: $\exists a \in \mathbb{Z}$ such that $\left((a^2 \not\equiv 0 \mod 4) \text{ and } (a^2 \not\equiv 1 \mod 4)\right)$.

(d) $\exists c \in \mathbb{R}_{>0}$ such that $\forall n \in \mathbb{N}\ (c^2 > n \implies c > n)$.

Solution: $\forall c \in \mathbb{R}_{>0}, \exists n \in \mathbb{N}$ such that $(c^2 > n \text{ and } c \leq n)$.

QUESTION 3. [**4 points**] Consider the following statement:

"For all sets A, B, and C, we have $A - (B \cup C) = (A - B) \cap (A - C)$."

Prove this statement is true or give a counterexample.

Solution: The statement is true. If A, B, and C are sets, we have

$$\begin{aligned}
x \in A - (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\
&\iff x \in A \text{ and } \neg(x \in B \cup C) \\
&\iff x \in A \text{ and } \neg(x \in B \text{ or } x \in C) \\
&\iff x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\
&\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\
&\iff x \in A - B \text{ and } x \in A - C \\
&\iff x \in (A - B) \cap (A - C).
\end{aligned}$$

QUESTION 4. [**4 points**] Consider the sequence $(x_n)_{n=1}^{\infty}$ defined recursively by

$$x_1 = 18,$$
$$x_{n+1} = x_n^2 + 6, \quad n \in \mathbb{N}.$$

Prove that x_n is divisible by 3 for all $n \in \mathbb{N}$.

Solution: We first consider the base case $n = 1$. Since

$$x_1 = 18 = 3 \cdot 6,$$

we see that x_1 is divisible by 3, proving the base case.

Now suppose the result is true for some $n \in \mathbb{N}$. Thus, there exists $k \in \mathbb{Z}$ such that $x_n = 3k$. Then we have

$$x_{n+1} = x_n^2 + 6 = (3k)^2 + 6 = 9k^2 + 6 = 3(3k^2 + 2).$$

Since $k \in \mathbb{Z}$, it follows that $3k^2 + 2 \in \mathbb{Z}$. Hence x_{n+1} is divisible by 3. This completes the proof of the induction step.

QUESTION 5. [**3 points**] Let

$$A = \{2a : a \in \mathbb{Z}\} \qquad \text{and} \qquad B = \{4b + 3 : b \in \mathbb{Z}\}.$$

Prove that $A \cap B = \varnothing$.

Solution: We prove the result by contradiction. Suppose there is an element $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus, there exist $a, b \in \mathbb{Z}$ such that

$$2a = x = 4b + 3.$$

This implies that $3 = 2a - 4b = 2(a - 2b)$, and so 3 is divisible by 2, which is a contradiction.

QUESTION 6. [**4 points**] Let $\sim$ be the relation on $\mathbb{R} \times \mathbb{R}$ defined by

$$(x, y) \sim (w, z) \iff x + z = y + w.$$

Prove that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $(x, y) \in \mathbb{R}\times\mathbb{R}$. Since $x+y = y+x$, we have $(x, y) \sim (x, y)$.

Symmetry: Suppose $(x, y), (w, z) \in \mathbb{R} \times \mathbb{R}$ satisfy $(x, y) \sim (w, z)$. Then $x + z = y + w$. Thus $w + y = z + x$, and so $(w, z) \sim (x, y)$.

Transitivity: Suppose $(x, y), (w, z), (a, b) \in \mathbb{R}\times\mathbb{R}$ satisfy $(x, y) \sim (w, z)$ and $(w, z) \sim (a, b)$. Then $x + z = y + w$ and $w + b = z + a$. Therefore,

$$x - y = w - z = a - b,$$

and so $x + b = y + a$. Hence $(x, y) \sim (a, b)$.

QUESTION 7. [**5 points**]

(a) Suppose $a \in \mathbb{N}$ satisfies $a \equiv 1 \mod 25$. Prove that $a \equiv 1 \mod 5$.

Solution: Since $a \equiv 1 \mod 25$, we have that $a-1$ is divisible by 25. Thus, there exists $k \in \mathbb{Z}$ such that

$$a - 1 = 25k = 5(5k).$$

Thus $a - 1$ is also divisible by 5, and so $a \equiv 1 \mod 5$.

(b) Is the statement

$$\forall a \in \mathbb{N}\,(a \equiv 1 \mod 5 \implies a \equiv 1 \mod 25)$$

true or false? Justify your answer.

Solution: No. For example, $6 \equiv 1 \mod 5$ (since $6 - 1 = 5$ is divisible by 5), but $6 \not\equiv 1 \mod 25$ (since $6 - 1 = 5$ is not divisible by 25).

(c) Find an integer n such that

$$70002 \le n \le 70009 \quad \text{and} \quad n^2 \equiv 1 \mod 7.$$

Solution: Fix the modulus 7. For $m \in \mathbb{Z}$, we have

$$[70000 + m]^2 = [m]^2.$$

Thus it suffices to find $m \in \mathbb{Z}$, $2 \le m \le 9$, such that $[m^2] = 1$. We compute:

$$[2^2] = [4], \quad [3^2] = [9] = [2], \quad [4^2] = [16] = [2], \quad [5^2] = [25] = [4],$$
$$[6^2] = [36] = [1], \quad [7^2] = [0]^2 = [0], \quad [8^2] = [1]^2 = [1], \quad [9^2] = [2]^2 = [4].$$

So $m = 6$ and $m = 8$ have the desired properties. Thus, $n = 70006$ and $n = 70008$ satisfy the conditions in the question.

(d) *Bonus*: Find a second integer n with the same properties.

Solution: See solution above.

QUESTION 8. [**7 points**] Consider the set

$$A = \left\{ 3 - \frac{6}{x} : x \in \mathbb{R},\ x \geq 1 \right\}.$$

(a) Find the supremum of A or prove that it does not exist.

Solution: We will show that $\sup A = 3$. Since

$$x \geq 1 \implies x > 0 \implies \frac{1}{x} > 0 \implies -\frac{6}{x} < 0 \implies 3 - \frac{6}{x} < 3,$$

we see that 3 is an upper bound for A. It remains to show that it is the *least* upper bound, which we do by contradiction. Suppose y is an upper bound for A with $y < 3$ (so $3 - y \in \mathbb{R}_{>0}$). We will arrive at a contradiction by finding $x \geq 1$ such that $3 - \frac{6}{x} > y$. Note that

$$3 - \frac{6}{x} > y \iff 3 - y > \frac{6}{x} \iff x > \frac{6}{3-y}.$$

Since $\mathbb{R}$ has no upper bound, we can find $x \in \mathbb{R}$ such that $\frac{6}{3-y} < x$ and $x \geq 1$. By the above, this x satisfies $3 - \frac{6}{x} < y$, and so y is not an upper bound for A.

(b) Find the largest element (i.e. maximum) of A or prove that it does not exist.

Solution: The set A does not have a largest element. If it did, such a largest element would have to be equal to $\sup A = 3$. But, by the above, we have $3 - \frac{6}{x} < 3$ for all $x \in \mathbb{R}_{>0}$. Thus $3 \notin A$.

(c) Find the smallest element (i.e. minimum) of A or prove that it does not exist.

Solution: We will show that $\min A = -3$. First note that

$$-3 = 3 - \frac{6}{1} \in A.$$

Also,

$$x \geq 1 \implies \frac{1}{x} \leq \frac{1}{1} = 1 \implies -\frac{6}{x} \geq -6 \implies 3 - \frac{6}{x} \geq -3.$$

Thus $a \geq -3$ for all $a \in A$. So $\min A = -3$.

(d) Find the infimum of A or prove that it does not exist.

Solution: We have $\inf A = \min A = -3$.

QUESTION 9. [**4 points**] Consider the function

$$f\colon \mathbb{R}_{>0} \to \mathbb{R}, \quad f(x) = 2 - \frac{8}{3x}.$$

(a) Is f injective? Justify your answer.

Solution: Suppose $x, y \in \mathbb{R}_{>0}$. We have

$$f(x) = f(y) \implies 2 - \frac{8}{5x} = 2 - \frac{8}{3y} \implies -\frac{8}{5x} = -\frac{8}{3y} \implies \frac{1}{x} = \frac{1}{y} \implies x = y.$$

Thus f is injective.

(b) What is the image of f? Write your answer as an interval. You do *not* need to justify your answer.

Solution: The image of f is $(-\infty, 2)$.

(c) Is f surjective?

Solution: No, f is not surjective, since its image does not equal its codomain.

Question 10. [**4 points**]

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.

Solution: The sequence $(x_k)_{k=1}^{\infty}$ converges to L if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |x_n - L| < \varepsilon.$$

(b) What is

$$\lim_{n\to\infty} \frac{1}{2n+3} \ ?$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences.

Solution: We will prove that the limit is 0. First note that, for $n \in \mathbb{N}$, we have

$$\left| \frac{1}{2n+3} - 0 \right| = \frac{1}{2n+3} \leq \frac{1}{2n} \leq \frac{1}{n}.$$

Let $\varepsilon > 0$. Since $\mathbb{N}$ has no upper bound, we can choose $N \in \mathbb{N}$ satisfying $N > \frac{1}{\varepsilon}$. Then, for $n \geq N$, we have

$$\left| \frac{1}{2n+3} - 0 \right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$