University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

Professor: Alistair Savage

Midterm Test

17 October 2016

Surname ____________________ First Name ____________________

Student # ____________________ DGD (1 or 2) ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	2	5	4	4	6	25
Grade							

QUESTION 1. [**4 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. Your grade on this question will then be rounded down to the nearest 0.5 points. You cannot receive a negative score on this question.

____ $(\exists x \in \mathbb{Z} \text{ such that})\,(\forall y \in \mathbb{Z})\, y - 5 = x$.

____ $(\forall x \in \mathbb{Z})\,(\exists y \in \mathbb{Z} \text{ such that})\, y - 5 = x$.

____ $(\exists a \in \mathbb{Z} \text{ such that})\,(\forall b \in \mathbb{N})\, a \neq b$.

____ $(\forall a \in \mathbb{Z})(\exists b \in \mathbb{Z} \text{ such that})\, ab > 0$.

____ $(-3 > 0) \implies (1 = 0)$.

____ $(\exists a \in \mathbb{Z} \text{ such that})\,(a < 0 \text{ and } a > 0)$.

____ $(\exists a \in \mathbb{Z} \text{ such that } a < 0) \text{ and } (\exists a \in \mathbb{Z} \text{ such that } a > 0)$.

____ $(1 > 0) \implies \big((\forall a \in \mathbb{Z})\, a > 0\big)$.

QUESTION 2. [**2 points**] Find the negations of the following statements. Simplify your answers as much as possible. In particular, your answers should not contain "not" or $\neg$.

(a) $(\forall a \in \mathbb{Z})\,(\exists b \in \mathbb{N} \text{ such that})\, 3a + b \neq 5$.

(b) $(\exists x \in \mathbb{N} \text{ such that})\,(\forall y \in \mathbb{Z})\,(x > y \text{ or } xy \neq 0)$.

(c) $P \implies \neg Q$.

(d) $(a \geq b)$ and no pigs are green.

QUESTION 3. Prove the following statements using *only* the axioms of $\mathbb{Z}$ (associativity, commutativity, distributivity, etc.). Justify each step by quoting (by name) the axiom you are using.

(a) **[2 points]** $\forall m \in \mathbb{Z},\ m \cdot 0 = 0$.

(b) **[3 points]** $\exists!\, n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Z},\ m \cdot (1 + n) = 0$.

Question 4.

(a) **[1 point]** Define the ordering $<$ on the integers. More precisely, complete the following sentence: "If $a, b \in \mathbb{Z}$, then $a < b$ if and only if...".

(b) **[3 points]** Using *only* properties of the natural numbers (closure under addition, closure under multiplication, etc.) and the definitions of $<$ and $\leq$, prove that if $a, b, c \in \mathbb{Z}$ such that $a < b$ and $c \leq 0$, then $bc \leq ac$.

QUESTION 5. [**4 points**] Let $(x_n)_{n=1}^{\infty}$ be the sequence defined by

$$x_0 = 1, \qquad x_1 = 2,$$
$$x_{n+2} = x_{n+1} + 3x_n, \quad n \geq 0.$$

Prove that $x_n > 2^n$ for all $n \geq 2$.

QUESTION 6.

(a) **[3 points]** Prove that

$$\{4n - 1 : n \in \mathbb{Z}\} = \{4m + 7 : m \in \mathbb{Z}\}.$$

(b) **[3 points]** Let A, B, and C be sets. Prove that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

You may use the fact that, if P, Q, and R are statements, then "P and (Q or R)" is equivalent to "(P and Q) or (P and R)".

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