University of Ottawa
Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs
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Midterm Test – Solutions
17 October 2016

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1 or 2) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	2	5	4	4	6	25
Grade							

QUESTION 1. **[4 points]** For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. Your grade on this question will then be rounded down to the nearest 0.5 points. You cannot receive a negative score on this question.

F $(\exists x \in \mathbb{Z} \text{ such that}) (\forall y \in \mathbb{Z})\, y - 5 = x$.

T $(\forall x \in \mathbb{Z}) (\exists y \in \mathbb{Z} \text{ such that})\, y - 5 = x$.

T $(\exists a \in \mathbb{Z} \text{ such that}) (\forall b \in \mathbb{N})\, a \neq b$.

F $(\forall a \in \mathbb{Z})(\exists b \in \mathbb{Z} \text{ such that})\, ab > 0$.

T $(-3 > 0) \implies (1 = 0)$.

F $(\exists a \in \mathbb{Z} \text{ such that}) (a < 0 \text{ and } a > 0)$.

T $(\exists a \in \mathbb{Z} \text{ such that } a < 0) \text{ and } (\exists a \in \mathbb{Z} \text{ such that } a > 0)$.

F $(1 > 0) \implies \big((\forall a \in \mathbb{Z})\, a > 0\big)$.

QUESTION 2. **[2 points]** Find the negations of the following statements. Simplify your answers as much as possible. In particular, your answers should not contain "not" or ¬.

(a) $(\forall a \in \mathbb{Z}) (\exists b \in \mathbb{N} \text{ such that})\, 3a + b \neq 5$.

Solution: $(\exists a \in \mathbb{Z} \text{ such that}) (\forall b \in \mathbb{N})\, 3a + b = 5$.

(b) $(\exists x \in \mathbb{N} \text{ such that}) (\forall y \in \mathbb{Z}) (x > y \text{ or } xy \neq 0)$.

Solution: $(\forall x \in \mathbb{N}) (\exists y \in \mathbb{Z} \text{ such that}) (x \leq y \text{ and } xy = 0)$.

(c) $P \implies \neg Q$.

Solution: P and Q.

(d) $(a \geq b)$ and no pigs are green.

Solution: $(a < b)$ or there exists a green pig.

Question 3. Prove the following statements using *only* the axioms of $\mathbb{Z}$ (associativity, commutativity, distributivity, etc.). Justify each step by quoting (by name) the axiom you are using.

(a) [**2 points**] $\forall m \in \mathbb{Z}$, $m \cdot 0 = 0$.

Solution: Suppose $m \in \mathbb{Z}$. Then we have

$$\begin{aligned}
m \cdot 0 &= m \cdot (0+0) && \text{(additive identity)}\\
\implies m \cdot 0 &= m \cdot 0 + m \cdot 0 && \text{(distributivity)}\\
\implies m \cdot 0 + \big(-(m \cdot 0)\big) &= (m \cdot 0 + m \cdot 0) + \big(-(m \cdot 0)\big) && \text{(replacement)}\\
\implies m \cdot 0 + \big(-(m \cdot 0)\big) &= m \cdot 0 + \big(m \cdot 0 + \big(-(m \cdot 0)\big)\big) && \text{(associativity of addition)}\\
0 &= m \cdot 0 + 0 && \text{(additive inverse)}\\
0 &= m \cdot 0 && \text{(additive identity)}
\end{aligned}$$

(b) [**3 points**] $\exists!\, n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Z}$, $m \cdot (1+n) = 0$.

Solution: *Existence:* Take $n = -1$. Then, for all $m \in \mathbb{Z}$, we have

$$\begin{aligned}
m \cdot (n+1) &= m \cdot (1 + (-1))\\
&= m \cdot 0 && \text{(additive inverse)}\\
&= 0. && \text{(part (a))}
\end{aligned}$$

So an n with the given property exists.

Uniqueness: Suppose $n \in \mathbb{Z}$ has the property that, for all $m \in \mathbb{Z}$, we have $m \cdot (1+n) = 0$. Then the equality holds when $m = 1$. Thus

$$\begin{aligned}
1 \cdot (1+n) &= 0\\
\implies 1+n &= 0 && \text{(multiplicative identity)}\\
\implies (-1) + (1+n) &= -1+0 && \text{(replacement)}\\
\implies (-1+1)+n &= -1+0 && \text{(associativity of addition)}\\
\implies 0+n &= -1+0 && \text{(additive inverse)}\\
\implies n+0 &= -1+0 && \text{(commutativity of addition)}\\
\implies n &= -1. && \text{(additive identity)}
\end{aligned}$$

Therefore -1 is the *only* n with the given property.

QUESTION 4.

(a) **[1 point]** Define the ordering $<$ on the integers. More precisely, complete the following sentence: "If $a, b \in \mathbb{Z}$, then $a < b$ if and only if...".

Solution: If $a, b \in \mathbb{Z}$, then $a < b$ if and only if $b - a \in \mathbb{N}$.

(b) **[3 points]** Using *only* properties of the natural numbers (closure under addition, closure under multiplication, etc.) and the definitions of $<$ and $\leq$, prove that if $a, b, c \in \mathbb{Z}$ such that $a < b$ and $c \leq 0$, then $bc \leq ac$.

Solution: Suppose $a, b, c \in \mathbb{Z}$ such that $a < b$ and $c \leq 0$. We split the proof into two cases: $c = 0$ and $c < 0$.

If $c = 0$, then $bc = 0 = ac$, and the result holds.

If $c < 0$, then we have $b - a \in \mathbb{N}$ and $-c = 0 - c \in \mathbb{N}$. Since $\mathbb{N}$ is closed under multiplication, we have $ac - bc = (b - a)(-c) \in \mathbb{N}$. Thus $bc < ac$.

QUESTION 5. [**4 points**] Let $(x_n)_{n=1}^{\infty}$ be the sequence defined by

$$x_0 = 1, \qquad x_1 = 2,$$
$$x_{n+2} = x_{n+1} + 3x_n, \quad n \geq 0.$$

Prove that $x_n > 2^n$ for all $n \geq 2$.

Solution: For $n \in \mathbb{N}$, let $P(n)$ be the statement

$$x_n > 2^n.$$

We have

$$x_2 = x_1 + 3x_0 = 2 + 3 \cdot 1 = 5 > 2^2 \quad \text{and} \quad x_3 = x_2 + 3x_1 = 5 + 3 \cdot 2 = 11 > 2^3,$$

and so $P(2)$ and $P(3)$ are true.

Now suppose that $n \geq 3$ and that $P(k)$ is true for $2 \leq k \leq n$. We will prove $P(n+1)$ is true. We have

$$x_{n+1} = x_n + 3x_{n-1} > 2^n + 3 \cdot 2^{n-1} = 2 \cdot 2^{n-1} + 3 \cdot 2^{n-1} = 5 \cdot 2^{n-1} > 4 \cdot 2^{n-1} = 2^{n+1},$$

where we have used the induction hypothesis that $P(n)$ and $P(n-1)$ are true. Thus $P(n+1)$ is true.

By strong induction, we conclude that $P(n)$ is true for all $n \geq 2$.

QUESTION 6.

(a) [**3 points**] Prove that

$$\{4n-1 : n \in \mathbb{Z}\} = \{4m+7 : m \in \mathbb{Z}\}.$$

Solution: Suppose $x \in \{4n-1 : n \in \mathbb{Z}\}$. Then there exists $n \in \mathbb{Z}$ such that $x = 4n-1$. So

$$x = 4n - 1 = 4n - 8 + 7 = 4(n-2) + 7.$$

Thus, $x = 4m+7$, where $m = n-2 \in \mathbb{Z}$. Thus $x \in \{4m+7 : m \in \mathbb{Z}\}$.

Now suppose $x \in \{4m+7 : m \in \mathbb{Z}\}$. Then there exists $m \in \mathbb{Z}$ such that $x = 4m+7$. So

$$x = 4m + 7 = 4m + 8 - 1 = 4(m+2) - 1.$$

Thus, $x = 4n-1$, where $n = m+2 \in \mathbb{Z}$. Therefore $x \in \{4n-1 : n \in \mathbb{Z}]$.

(b) [**3 points**] Let A, B, and C be sets. Prove that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

You may use the fact that, if P, Q, and R are statements, then "P and (Q or R)" is equivalent to "(P and Q) or (P and R)".

Solution: We have

$$\begin{aligned} & x \in A - (B \cap C) \\ \iff & x \in A \text{ and } x \notin B \cap C \\ \iff & x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ \iff & (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ \iff & x \in A - B \text{ or } x \in A - C \\ \iff & x \in (A - B) \cup (A - C). \end{aligned}$$