University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

Professor: Alistair Savage

Final Exam

December 2016

Surname ________________________ First Name ________________________

Student # ______________________ Seat # ________________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	Total
Maximum	4	3	3	2	3	3	3	7	3	4	35
Grade											

Question 1. [**4 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. (You will not lose points for incorrect answers.)

____ $(\forall x \in \mathbb{R})\,(\exists n \in \mathbb{Z}$ such that$)\, n \geq x$.

____ $(\forall n \in \mathbb{Z})\,(\exists x \in \mathbb{R}$ such that$)\, x \geq n$.

____ $(\exists n \in \mathbb{N}$ such that $n < 0) \implies (1 = 0)$.

____ If $f\colon A \to B$ and $g\colon B \to C$ are functions and f is injective, then $g \circ f$ is injective.

____ If $f\colon A \to B$ and $g\colon B \to C$ are functions and $g \circ f$ is surjective, then g is surjective.

____ If A is a nonempty set that has a supremum, then A has a greatest element.

____ If A is a nonempty set that has a smallest element, then A has an infimum.

____ It is possible for a sequence $(x_n)_{n=1}^{\infty}$ to have limits L and L' with $L \neq L'$.

QUESTION 2. [**3 points**] Suppose A, B, and C are sets. Prove that

$$(A \times B) \cap (C \times B) = (A \cap C) \times B.$$

QUESTION 3. [**3 points**] Prove that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1} \quad \text{for all } n \in \mathbb{N}.$$

QUESTION 4. [**2 points**] Let

$$A = \{16a + 11 : a \in \mathbb{Z}\} \qquad \text{and} \qquad B = \{8b + 3 : b \in \mathbb{Z}\}.$$

Prove that $A \subseteq B$.

QUESTION 5. [**3 points**] Let $\sim$ be the relation on $\mathbb{R}^2 - \{(0,0)\}$ defined by

$$(x, y) \sim (w, z) \iff xz = yw.$$

Prove that $\sim$ is an equivalence relation.

QUESTION 6. [**3 points**] Suppose $n \in \mathbb{Z}$, $n \geq 2$.

(a) Define the relation on $\mathbb{Z}$ of equivalence modulo n. More precisely, complete the following sentence: "For $a, b \in \mathbb{Z}$, $a \equiv b \mod n$ if and only if. . .."

(b) Prove that $(n+1)^2 \equiv 1 \mod n$.

QUESTION 7. [**3 points**]

(a) State Fermat's Little Theorem.

(b) Suppose $a \in \mathbb{Z}$, and p is a prime number. Prove that

$$(a + a^p)^p \equiv 2a \mod p.$$

QUESTION 8. [**7 points**] Consider the set

$$A = \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\}.$$

(a) Find the infimum of A or prove that it does not exist.

(b) Find the smallest element of A or prove that it does not exist.

(c) Find the greatest element of A or prove that it does not exist.

(d) Find the supremum of A or prove that it does not exist.

QUESTION 9. [**3 points**] Consider the function

$$f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \quad f(x) = 6x + 5.$$

(a) Prove that f is injective.

(b) What is the image of f? Write your answer as an interval. You do *not* need to justify your answer.

(c) Is f surjective?

QUESTION 10. [**4 points**]

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.

(b) What is

$$\lim_{n\to\infty}\left(2-\frac{3}{n}\right)\ ?$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences.

This page is intentionally left blank. You may use it as scrap paper.