University of Ottawa
Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs
Professor: Alistair Savage

Final Exam – Solutions
December 2016

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	Total
Maximum	4	3	3	2	3	3	3	7	3	4	35
Grade											

Question 1. [**4 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. (You will not lose points for incorrect answers.)

T $(\forall x \in \mathbb{R})\,(\exists n \in \mathbb{Z} \text{ such that})\, n \geq x$.

T $(\forall n \in \mathbb{Z})\,(\exists x \in \mathbb{R} \text{ such that})\, x \geq n$.

T $(\exists n \in \mathbb{N} \text{ such that } n < 0) \implies (1 = 0)$.

F If $f\colon A \to B$ and $g\colon B \to C$ are functions and f is injective, then $g \circ f$ is injective.

T If $f\colon A \to B$ and $g\colon B \to C$ are functions and $g \circ f$ is surjective, then g is surjective.

F If A is a nonempty set that has a supremum, then A has a greatest element.

T If A is a nonempty set that has a smallest element, then A has an infimum.

F It is possible for a sequence $(x_n)_{n=1}^{\infty}$ to have limits L and L' with $L \neq L'$.

QUESTION 2. [**3 points**] Suppose A, B, and C are sets. Prove that

$$(A \times B) \cap (C \times B) = (A \cap C) \times B.$$

Solution: We have

$$\begin{aligned}
(x,y) \in (A \times B) \cap (C \times B) &\iff (x,y) \in A \times B \text{ and } (x,y) \in C \times B \\
&\iff (x \in A \text{ and } y \in B) \text{ and } (x \in C \text{ and } y \in B) \\
&\iff x \in A \text{ and } y \in B \text{ and } x \in C \\
&\iff x \in A \cap C \text{ and } y \in B \\
&\iff (x,y) \in (A \cap C) \times B.
\end{aligned}$$

Therefore the given sets are equal.

QUESTION 3. [**3 points**] Prove that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Solution: We first consider the base case $n = 1$. We have

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = 1 - \frac{1}{1+1},$$

proving the base case.

Now suppose the result is true for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{1}{(n+1)(n+2)} + \sum_{i=1}^{n} \frac{1}{i(i+1)} &= \frac{1}{(n+1)(n+2)} + 1 - \frac{1}{n+1} \\ &= 1 + \frac{1}{n+1}\left(\frac{1}{n+2} - 1\right) = 1 - \frac{1}{n+1}\frac{n+1}{n+2} = 1 - \frac{1}{n+2},\end{aligned}$$

where we used the induction hypothesis in the second equality. This completes the proof of the induction step.

QUESTION 4. [**2 points**] Let

$$A = \{16a + 11 : a \in \mathbb{Z}\} \qquad \text{and} \qquad B = \{8b + 3 : b \in \mathbb{Z}\}.$$

Prove that $A \subseteq B$.

Solution: Suppose $x \in A$. Then there exists $a \in \mathbb{Z}$ such that $x = 16a + 11$. So

$$x = 16a + 11 = 16a + 8 + 3 = 8(2a + 1) + 3 = 8b + 3,$$

where $b = 2a + 1 \in \mathbb{Z}$. Thus $x \in B$. Therefore $A \subseteq B$.

QUESTION 5. [**3 points**] Let $\sim$ be the relation on $\mathbb{R}^2 - \{(0,0)\}$ defined by

$$(x,y) \sim (w,z) \iff xz = yw.$$

Prove that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$. Since $xy = yx$, we have $(x,y) \sim (x,y)$.

Symmetry: Suppose $(x,y),(w,z) \in \mathbb{R}^2 - \{(0,0)\}$ satisfy $(x,y) \sim (w,z)$. Then $xz = yw$. Thus $wy = yw = xz = zx$, and so $(w,z) \sim (x,y)$.

Transitivity: Suppose $(x,y),(w,z),(a,b) \in \mathbb{R}^2 - \{(0,0)\}$ satisfy $(x,y) \sim (w,z)$ and $(w,z) \sim (a,b)$. Then $xz = yw$ and $wb = za$. If $z \neq 0$, then

$$xb = \frac{xbz}{z} = \frac{(xz)b}{z} = \frac{(yw)b}{z} = \frac{y(wb)}{z} = \frac{yza}{z} = ya.$$

Thus $(x,y) \sim (a,b)$. On the other hand, if $z = 0$, then $w \neq 0$ (since $(w,z) \neq (0,0)$), and so

$$xb = \frac{xbw}{w} = \frac{x(wb)}{w} = \frac{x(za)}{w} = \frac{(xz)a}{w} = \frac{ywa}{w} = ya.$$

QUESTION 6. [**3 points**] Suppose $n \in \mathbb{Z}$, $n \geq 2$.

(a) Define the relation on $\mathbb{Z}$ of equivalence modulo n. More precisely, complete the following sentence: "For $a, b \in \mathbb{Z}$, $a \equiv b \mod n$ if and only if. . . ."

Solution: For $a, b \in \mathbb{Z}$, $a \equiv b \mod n$ if and only if $a - b$ is divisible by n

(b) Prove that $(n+1)^2 \equiv 1 \mod n$.

Solution: We have

$$(n+1)^2 - 1 = n^2 + 2n + 1 - 1 = n^2 + 2n = n(n+2)$$

is divisible by n. Therefore $(n+1)^2 \equiv 1 \mod n$.

QUESTION 7. [**3 points**]

(a) State Fermat's Little Theorem.

Solution: If $m \in \mathbb{Z}$ and p is prime, then $m^p \equiv m \mod p$.

(b) Suppose $a \in \mathbb{Z}$, and p is a prime number. Prove that

$$(a + a^p)^p \equiv 2a \mod p.$$

Solution: Apply Fermat's Little Theorem twice, we have (working modulo p)

$$(a + a^p)^p \equiv a + a^p \equiv a + a \equiv 2a.$$

QUESTION 8. [**7 points**] Consider the set

$$A = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}.$$

(a) Find the infimum of A or prove that it does not exist.

Solution: We will show that $\inf A = 1$. Since

$$n \in \mathbb{N} \implies n > 0 \implies \frac{1}{n} > 0 \implies 1 + \frac{1}{n} > 1,$$

we see that 1 is a lower bound for A. It remains to show that it is the *greatest* lower bound, which we do by contradiction. Suppose y is a lower bound for A with $1 < y$. We will arrive at a contradiction by finding $n \in \mathbb{N}$ such that $1 + \frac{1}{n} < y$. Note that

$$1 + \frac{1}{n} < y \iff \frac{1}{n} < y - 1 \iff n > \frac{1}{y-1}.$$

Since $\mathbb{N}$ has no upper bound, we can always find $n \in \mathbb{N}$ satisfying $n > \frac{1}{y-1}$. By the above, this n satisfies $1 + \frac{1}{n} < y$, and so y is not a lower bound for A.

(b) Find the smallest element of A or prove that it does not exist.

Solution: The set A does not have a smallest element. If it did, such a smallest element would have to be equal to $\inf A = 1$. But, by the above, we have $1 + \frac{1}{n} > 1$ for all $n \in \mathbb{N}$. Thus $1 \notin A$.

(c) Find the greatest element of A or prove that it does not exist.

Solution: We will show that $\max A = 2$. First note that

$$2 = 1 + \frac{1}{1} \in A.$$

Also,

$$n \in \mathbb{N} \implies n \geq 1 \implies \frac{1}{n} \leq \frac{1}{1} = 1 \implies 1 + \frac{1}{n} \leq 2.$$

Thus $a \leq 2$ for all $a \in A$. So $\max A = 2$.

(d) Find the supremum of A or prove that it does not exist.

Solution: We have $\sup A = \max A = 2$.

QUESTION 9. [**3 points**] Consider the function

$$f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \quad f(x) = 6x + 5.$$

(a) Prove that f is injective.

Solution: Suppose $x, y \in \mathbb{R}_{>0}$. We have

$$f(x) = f(y) \implies 6x + 5 = 6y + 5 \implies 6x = 6y \implies x = y.$$

Thus f is injective.

(b) What is the image of f? Write your answer as an interval. You do *not* need to justify your answer.

Solution: The image of f is $(5, \infty)$.

(c) Is f surjective?

Solution: No, f is not surjective, since its image does not equal its codomain.

QUESTION 10. [**4 points**]

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.

Solution: The sequence $(x_k)_{k=1}^{\infty}$ converges to L if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \; |x_n - L| < \varepsilon.$$

(b) What is

$$\lim_{n\to\infty} \left(2 - \frac{3}{n}\right) ?$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences.

Solution: We will prove that the limit is 2. Let $\varepsilon > 0$. Since $\mathbb{N}$ has no upper bound, we can choose $N \in \mathbb{N}$ satisfying $N > 3/\varepsilon$. Then, for $n \geq N$, we have

$$\left| \left(2 - \frac{3}{n}\right) - 2 \right| = \frac{3}{n} \leq \frac{3}{N} < \frac{3}{3/\varepsilon} = \varepsilon.$$