MAT 1302B – Mathematical Methods II

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Winter 2015 – Lecture 21 (review)

Announcements

- Math help centre (Marion Hall, Room 021) exam period hours:
  - April 13–24: 10am–5pm Monday to Friday (with some exceptions)

About the exam

Format:

- No notes, books, calculators, etc.
- You may write in pen or pencil.
- Answer only questions: only your final answer will be graded.
- Long answer questions: you must show your work and justify your answers.

Important!

- Read the questions carefully before starting to answer them.
- After completing each question, reread the question to make sure you answered it.
- Don't rush—this might cause you to make mistakes.
- Use any extra time to check over your answers.

Best way to study: Do lots of questions (recommended exercises, old exams, etc.) and go over your mistakes on midterms and assignments.

Reminder: Computation of grades

- You must pass the final exam to pass the course.

- If you pass the final exam, your grade will be computed as follows:
  - Each midterm worth 15% (45% total).
  - Final exam worth 55%.

- Your lowest midterm grade will automatically be replaced by your grade on the final, if this is to your advantage.
  - If you missed a midterm, this is the one that will be replaced.

Review topics (as selected by students)

1. subspaces (subspace test)
2. theory questions (true/false)
3. eigenvectors/eigenvalues and the characteristic equation
4. diagonalization
5. linear independence
6. determinants
7. network flow

Note: We are not reviewing all the topics!

Subspaces

Definition (Subspace)

A subspace of $\mathbb{R}^n$ is any subset H of $\mathbb{R}^n$ that satisfies the following 3 conditions:

1. $\vec{0} \in H$.
2. If $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$.
3. If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$.

Important examples of subspaces

1. The set $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.
2. $\mathbb{R}^n$ is a subspace of itself.
3. Given vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, their span

   $$\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$$

   is a subspace of $\mathbb{R}^n$.
4. Column spaces: If A is an $m \times n$ matrix, then $\operatorname{Col} A$ is the span of the columns of A and hence is a subspace of $\mathbb{R}^m$.

   $\operatorname{Col} A$ is also the set of $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ has a solution.
5. Null spaces / set of solutions to a homogeneous system: If A is an $m \times n$ matrix, $\operatorname{Nul} A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$.
6. Eigenspaces: If λ is an eigenvalue of A, the corresponding eigenspace is the set of solutions to

   $$A\vec{x} = \lambda\vec{x} \qquad \text{or} \qquad (A - \lambda I)\vec{x} = \vec{0}$$

   (example of a null space).

Subspaces – examples

Which of the following are subspaces of $\mathbb{R}^n$ for the given n?

1.
$$H = \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \middle| \, x = 2y, \right\}, \quad n = 4$$

   Answer: YES. This is the set of solutions to a homogeneous system.

2.
$$K = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \, a + 2b = 3c, \; b = d \right\}, \quad n = 4$$

   Answer: YES. Again, this is the set of solutions to a homogeneous system.

Subspaces – examples

3.

$$L = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \, a + 2b = 3c, \; b = d + 1 \right\}, \quad n = 4$$

Answer: NO. $\vec{0} \notin L$ since we cannot have both b and d be zero.

4.

$$M = \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \middle| \, x \leq 0 \right\}, \quad n = 4$$

Answer: NO. M is not closed under scalar multiplication. For example

$$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \in M \quad \text{but} \quad (-1) \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \notin M.$$

Subspaces – examples

5. $$\{(a, b, 3) \mid a, b \in \mathbb{R}\}, \quad n = 3$$

   Answer: NO. $\vec{0}$ is not in the set.

6. $$\{(0, 0)\}, \quad n = 3$$

   Answer: NO. The set is not a subset of $\mathbb{R}^3$.

7. $$\{(0, 0)\}, \quad n = 2$$

   Answer: YES.

8. $$\{(a, b, a, -b) \mid a, b \in \mathbb{R}\}, \quad n = 4$$

   Answer: YES. This set is equal to

   $$\operatorname{Span}\{(1, 0, 1, 0), (0, 1, 0, -1)\}.$$

Strategy for determining if a set is a subspace or not

Important: There is no algorithm (or recipe) for completely answering these types of questions. However, here are some suggestions for things to try.

1. Make sure the dimensions are right. If you're looking for a subspace of $\mathbb{R}^n$, your vectors should have n entries each.
2. Is $\vec{0}$ in the set? If not, you're done – the set is not a subspace.
3. Can you identify the set as one of the standard examples of a subspace? That is, can you write it as
   - the span of some set of vectors, or the column space of a matrix?
   - the solution set to a system of homogeneous equations (equivalently, the null space of some matrix)?
   - an eigenspace?

   If yes, you're done – the set is a subspace.
4. Check the addition and scalar multiplication axioms for a subspace directly. Can you find
   - a vector in your set such that some scalar multiple of it is not in your set?
   - two vectors in your set whose sum is not in your set?

Theory questions

Some of the short answer questions will cover theory.

The best way to study for these is to:

- Review the theorems (and propositions, corollaries, etc.) and definitions in the slides.
- Go through the short answer questions on old exams—questions tend to repeat!

Eigenvectors, eigenvalues, and eigenspaces

Definition (eigenvectors and eigenvalues)

Suppose A is a square matrix. If $\vec{x}$ is a nonzero vector and λ is a scalar such that

$$A\vec{x} = \lambda\vec{x}, \qquad \vec{x} \neq \vec{0},$$

then

- λ is an eigenvalue of A, and
- $\vec{x}$ is an eigenvector of A (an eigenvector corresponding to the eigenvalue λ).

If λ is an eigenvalue, then the set of solutions to $A\vec{x} = \lambda\vec{x}$ (or $(A - \lambda I)\vec{x} = \vec{0}$) is the eigenspace corresponding to λ.

Note: Since the eigenspaces are null spaces of $A - \lambda I$, they are subspaces.

Finding eigenvalues and eigenvectors/eigenspaces

Procedure for finding eigenvalues, eigenvectors and eigenspaces

1. To find the eigenvalues, find the solutions to the characteristic equation
$$\det(A - \lambda I) = 0.$$
2. For each each eigenvalue, solve the equation
$$(A - \lambda I)\vec{x} = \vec{0}$$
to find the corresponding eigenspace.
3. The nonzero vectors in each eigenspace are the eigenvectors corresponding to the given eigenvalue.

Example (Lay, 3rd Edition, Page 326, # 13)

Find the eigenvalues and corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

Solution: We first find the characteristic polynomial.

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix} \\
&= (2-\lambda)\big((3-\lambda)(2-\lambda) - 2\big) - 2\big(2 - \lambda - 1\big) \\
&\quad + (-1)\big(-2 + (3-\lambda)\big) \\
&= (2-\lambda)(4 - 5\lambda + \lambda^2) - 2(1-\lambda) - (1-\lambda) \\
&= (2-\lambda)(1-\lambda)(4-\lambda) - 3(1-\lambda) \\
&= (1-\lambda)\big((2-\lambda)(4-\lambda) - 3\big) = (1-\lambda)(5 - 6\lambda + \lambda^2) \\
&= (1-\lambda)(1-\lambda)(5-\lambda)
\end{aligned}$$

Example (cont.)

Thus, the eigenvalues are 1 (with multiplicity 2) and 5 (with multiplicity 1).

Now we find the eigenspace for each eigenvalue.

For $\lambda = 1$, we solve

$$\left[\begin{array}{c|c} A - I & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ll} x_1 & = -2x_2 + x_3 \\ x_2 & \text{free} \\ x_2 & \text{free} \end{array} \implies \vec{x} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ x_2, x_3 \in \mathbb{R}$$

Therefore, the eigenspace corresponding to the eigenvalue 1 is

$$\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Example (cont.)

For $\lambda = 5$, we solve

$$\left[\begin{array}{c|c} A - 5I & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -3 & 2 & -1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & -2 & -3 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ll} x_1 & = -x_3 \\ x_2 & = -x_3 \\ x_3 & \text{free} \end{array} \implies \vec{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

Therefore, the eigenspace corresponding to the eigenvalue 5 is

$$\operatorname{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Example (cont.)

Check your answer!

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad \checkmark$$

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \checkmark$$

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \checkmark$$

Diagonalization: Example

Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

from the previous example, if possible. That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (or $D = P^{-1}AP$).

Solution: We found that the eigenvalues were 1 (multiplicity 2) and 5 (multiplicity 1).

A basis for the eigenspace of eigenvalue 1 was $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$,

and a basis for the eigenspace of eigenvalue 5 was $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Example (cont.)

For each eigenvalue, the multiplicity is equal to the dimension of the corresponding eigenspace. Thus the matrix is diagonalizable.

We form D by placing the eigenvalues on the diagonal (according to their multiplicity):

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

We form P by placing the eigenvectors as the columns in the same order as the eigenvalues:

$$P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Example (cont.)

Check your answer!

$$AP = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -5 \\ 1 & 0 & -5 \\ 0 & 1 & 5 \end{bmatrix}$$

$$PD = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -5 \\ 1 & 0 & -5 \\ 0 & 1 & 5 \end{bmatrix}$$

Thus $AP = PD$. ✓

Finding roots of a characteristic polynomial

Quadratic polynomials ($a\lambda^2 + b\lambda + c$): Two methods.

- Factor. If $a = 1$, find two numbers whose product is c and whose sum is b.

  Example: $\lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3)$.

- Quadratic equation.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Finding roots of a characteristic polynomial

Polynomials of higher degrees (cubics, quartics, etc.):

Usually, if you're smart when computing the determinant, some of the factoring is already done.

Example:

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 0 & 0 \\ 5 & -5-\lambda & 6 \\ -7 & -1 & -\lambda \end{vmatrix} &= (2-\lambda)\begin{vmatrix} -5-\lambda & 6 \\ -1 & -\lambda \end{vmatrix} \\ &= (2-\lambda)(\lambda^2+5\lambda+6) \\ &= (2-\lambda)(\lambda+2)(\lambda+3) \end{aligned}$$

Linear independence

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent if the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0} \tag{1}$$

has only the trivial solution.

If (1) has a nontrivial solution, then the vectors are linearly dependent, and any nontrivial solution is a dependence relation.

In general, to see if the vectors are linearly independent or not, you solve the equation (1) to see if there are free variables.

Shortcuts (that only work in some special cases):

- Any list containing the zero vector is linearly dependent.
- For lists with two vectors: $\vec{v}_1$, $\vec{v}_2$ are linearly independent iff neither is a scalar multiple of the other (i.e. they are not parallel).

Linear independence: example

Question: Are the vectors

$$(1, 2, -2, 4, 2),\ (0, 1, 0, 2, 3),\ (0, 0, 1, 1, 2),\ (-3, -5, 4, -12, -7)$$

linearly independent? If not, find a linear dependence relation.

Solution: We row reduce:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 2 & 1 & 0 & -5 & 0 \\ -2 & 0 & 1 & 4 & 0 \\ 4 & 2 & 1 & -12 & 0 \\ 2 & 3 & 2 & -7 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a free variable, the vectors are not linearly independent. The general solution is

$$x_1 = 3x_4,\ x_2 = -x_4,\ x_3 = 2x_4, \quad x_4 \in \mathbb{R}.$$

Taking $x_4 = 1$ gives the linear dependence relation

$$3(1, 2, -2, 4, 2) - (0, 1, 0, 2, 3) + 2(0, 0, 1, 1, 2) + (-3, -5, 4, -12, -7) = \vec{0}.$$

Determinants

Methods of computing determinants:

- Row/column expansion (choose rows or columns with lots of zeros if possible).
- Row/column operations:
  - Row/column replacement (adding a multiple of one row/column to another): no change in the determinant.
  - Row/column interchange: determinant multiplied by -1.
  - Multiplying a row/column by c: determinant multiplied by c.
- Combination of the two approaches.

Determinants

Expressions involving several matrices: Suppose A, B, C and D are all 2×2 matrices with

$$\det A = 3, \quad \det B = -1, \quad \det C = 2, \quad \det D \neq 0.$$

What is $\det(3A^2D^4B^TBCB^{-1}(D^T)^{-1}D^{-3})$?

Since the matrices are 2×2, we have

$$\begin{aligned}
&\det(3A^2D^4B^TBCB^{-1}(D^T)^{-1}D^{-3}) \\
&=3^2(\det A)^2(\det D)^4(\det B^T)(\det B)(\det C)(\det B)^{-1}(\det D^T)^{-1}(\det D)^{-3} \\
&=3^2(\det A)^2(\det D)^4(\det B)(\det B)(\det C)(\det B)^{-1}(\det D)^{-1}(\det D)^{-3} \\
&=3^2(\det A)^2(\det B)(\det C) \\
&=3^2 \cdot 3^2 \cdot (-1) \cdot 2 \\
&=-162.
\end{aligned}$$

Note: We didn't need to know $\det D$!

Determinants

Suppose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5a & b-2c & c \\ 5d & e-2f & f \\ 5g & h-2i & i \end{bmatrix}.$$

If $\det A = -2$, what is $\det B$?

Solution: We obtain the second matrix from the first by

- multiplying the first column by 5 (which multiplies the determinant by 5), and
- adding -2 times the third column to the second column (which leaves the determinant unchanged).

Thus, $\det B = (-2) \cdot 5 = -10$.

Remember:

- If we had swapped two columns, this would have multiplied the determinant by -1.
- We also know how row operations affect the determinant.

Network flow: variable ranges

Question: Suppose we've solved a network flow problem, where variables represent one-directional flows, and the general solution is:

$$\begin{aligned} x_1 &= 250 - x_5 \\ x_2 &= 600 - 6x_5 \\ x_3 &= 100 + 2x_5 \\ x_4 &= x_5 - 50 \\ x_5 &\text{ free} \end{aligned}$$

What are the possible ranges of all the variables?

Solution: The fact that each variable has to be ≥ 0 puts some restrictions on the values of the free variable:

- $x_1 \geq 0 \implies x_5 \leq 250$
- $x_2 \geq 0 \implies 600 - 6x_5 \geq 0 \implies x_5 \leq 100$
- $x_3 \geq 0$ imposes no restriction on x_5 (since we already have $x_5 \geq 0$)
- $x_4 \geq 0 \implies x_5 \geq 50$

Thus, we must have $50 \leq x_5 \leq 100$.

Network flow: variable ranges (cont.)

We have:

$$
\begin{aligned}
x_1 &= 250 - x_5 \\
x_2 &= 600 - 6x_5 \\
x_3 &= 100 + 2x_5 \\
x_4 &= x_5 - 50 \\
50 \le x_5 &\le 100.
\end{aligned}
$$

So the ranges on the other variables are:

$$150 \le x_1 \le 200,\ 0 \le x_2 \le 300,\ 200 \le x_3 \le 300,\ 0 \le x_4 \le 50.$$

Good luck on the exam!