

MAT 1302B – Mathematical Methods II

Alistair Savage

Mathematics and Statistics
University of Ottawa

Winter 2015 – Lecture 19

These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Announcements

Review class: April 14 (10:00am)

- The last class is open for review.
- We can go over theory, do extra examples, or go through an old exam.
- Think of topics that you'd like to go over – I'll collect requests next class.

Third Midterm

- Handed back in DGDs this week.

Eigenvectors, eigenvalues, and eigenspaces

Definition (eigenvectors and eigenvalues)

Suppose A is a square matrix. If \vec{x} is a nonzero vector and λ is a scalar such that

$$A\vec{x} = \lambda\vec{x}, \quad \vec{x} \neq \vec{0}$$

then

- λ is an **eigenvalue** of A , and
- \vec{x} is an **eigenvector** of A (an eigenvector corresponding to the eigenvalue λ).

If λ is an eigenvalue, then the set of solutions to $A\vec{x} = \lambda\vec{x}$ (or $(A - \lambda I)\vec{x} = \vec{0}$) is the **eigenspace** corresponding to λ .

Note: Since the eigenspaces are null spaces of $A - \lambda I$, they are subspaces.

Recap: finding eigenvalues and eigenvectors/eigenspaces

Procedure for finding eigenvalues, eigenvectors and eigenspaces

- 1 To find the eigenvalues, find the solutions to the characteristic equation

$$\det(A - \lambda I) = 0.$$

- 2 For each each eigenvalue, solve the equation

$$(A - \lambda I)\vec{x} = \vec{0}$$

to find the corresponding eigenspace.

- 3 The nonzero vectors in each eigenspace are the eigenvectors corresponding to the given eigenvalue.

Diagonal matrices

Recall: a **diagonal matrix** is a square matrix whose only nonzero entries lie on the main diagonal.

Examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Diagonal matrices are easy to work with for many reasons:

- They are both upper and lower triangular.
- Their eigenvalues are just the entries on the diagonal.
- Their determinant is simply the product of the entries on the diagonal.
- Easy to see if they are invertible (and find inverses).
- Easy to multiply (just multiply corresponding diagonal entries).

Example

Powers of arbitrary matrices can be difficult to compute:

$$A = \begin{bmatrix} 2 & 10 & -12 & -1 \\ -3 & -1 & 6 & 3 \\ 4 & 8 & 5 & 4 \\ 7 & 5 & 7 & 4 \end{bmatrix}, \quad A^4 \text{ is a lot of work to compute.}$$

Powers (or products) of diagonal matrices are relatively simple:

$$D = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D^4 =$$

Goal: Somehow relate arbitrary matrices to diagonal matrices.

Example

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}, \quad \text{so} \quad P^{-1} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$$

$$\text{Then } A \stackrel{\text{def}}{=} PDP^{-1} = \begin{bmatrix} 23 & -42 \\ 12 & -22 \end{bmatrix}.$$

Suppose we want to compute A^4 . Directly, this would be a lot of work.

However, notice

$$A^4 =$$

Therefore

$$A^4 =$$

For higher powers, save even more work: A^k

Some definitions

Definition (similar matrices)

If $A = PBP^{-1}$ for some invertible matrix P , we say A and B are **similar**.

Goal

For a matrix A , try to find a diagonal matrix similar to A .

Definition (diagonalizable)

If A is similar to a diagonal matrix, we say A is **diagonalizable**.

So A is diagonalizable if there is some diagonal matrix D and some invertible matrix P such that $A = PDP^{-1}$.

To **diagonalize** a matrix A means to find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Note: Not all matrices are diagonalizable!

Example

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$. Let's find the eigenvalues of A .

Now we find the eigenspace for each eigenvalue.

Example (cont.)

Example (cont.)

Example (cont.)

Suppose we now want to compute A^{100} .

Directly computing

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}^{100}$$

would be a **lot** of work (99 matrix multiplications).

However, since we've diagonalized A , we can compute this in an easier way:

Another example

Suppose

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Then

$$\det(A - \lambda I) = (2 - \lambda)^2(8 - \lambda). \quad (\text{exercise})$$

Another example (cont.)

Another example (cont.)

Another example (cont.)

Comments

- 1 Can reorder the eigenvalues/eigenvectors – just make sure the order of eigenvalues matches the order of eigenvectors.

Example: In the previous example, we had

$$P = \quad , \quad D =$$

but we could have taken

$$P = \quad , \quad D =$$

Comments (cont.)

- 2 For a given eigenvalue, can choose **any** basis of eigenvectors for the corresponding eigenspace.
- 3 A matrix is diagonalizable iff the dimension of **each** eigenspace equals the multiplicity of the corresponding eigenvalue (the dimension will always be \leq the multiplicity).

Equivalently, an $n \times n$ matrix is diagonalizable iff the sum of the dimensions of the eigenspaces is n .

Definition (geometric multiplicity)

The **geometric multiplicity** of an eigenvalue is the dimension of the corresponding eigenspace.

Warnings!

“Diagonalizable” \neq “invertible”!

Diagonalizable is **not** the same thing as invertible!

A matrix can be invertible but not diagonalizable or diagonalizable but not invertible.

However...remember that a matrix is invertible iff it does not have zero as an eigenvalue. So if you've diagonalized a matrix (hence found its eigenvalues), you can easily tell if it's invertible or not.

“Similar” \neq “row equivalent”!

Similar is **not** the same this as row equivalent!

A is **row equivalent** to B if you can get from A to B via a sequence of row operations.

A is **similar** to B if there is some invertible matrix P such that $A = PBP^{-1}$.

Example

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Characteristic polynomial: $\det(A - \lambda I) = (3 - \lambda)^2(2 - \lambda)$

Note

Remember that λ is an eigenvalue iff $(A - \lambda I)\vec{x} = \vec{0}$ has some nontrivial solution (i.e. there is at least one free variable in the solution set).

Thus, the dimension of the eigenspace corresponding to any eigenvalue is always **at least** one.

Theorem

An $n \times n$ matrix with n distinct eigenvalues is always diagonalizable.

Reason

- An $n \times n$ matrix has n eigenvalues counting multiplicity.
- If it has n distinct eigenvalues, each one must have multiplicity one.
- Since the dimension of each eigenspace is at least one and can be at most one (since the dimension of the eigenspace is always \leq the multiplicity), it is **exactly** one.
- So the matrix is diagonalizable.

Procedure for diagonalizing matrices

Suppose we want to diagonalize the $n \times n$ matrix A .

- 1 Find the eigenvalues of A by solving $\det(A - \lambda I) = 0$.
- 2 For each eigenvalue, find the corresponding eigenspace:
 - ▶ Row reduce $[A - \lambda I \mid \vec{0}]$.
 - ▶ Write the solution in vector parametric notation:

$$\vec{x} = x_{i_1} \vec{v}_1 + x_{i_2} \vec{v}_2 + \cdots + x_{i_k} \vec{v}_k$$

- ▶ Eigenspace = $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

If dimension $k <$ the multiplicity of the eigenvalue, then A is not diagonalizable.

- 3 Construct P from the eigenvectors found in Step 2:

$$P = [\vec{v}_1 \quad \vec{v}_2 \quad \cdots]$$

- 4 Construct D from the eigenvalues: $D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$ where

$\lambda_i =$ eigenvalue of \vec{v}_i .

- 5 Check by verifying $A = PDP^{-1}$ or $AP = PD$ (easier, since don't need to compute P^{-1}).

The Diagonalization Theorem

Definition (eigenvector basis)

If A is an $n \times n$ matrix, then an **eigenvector basis** for A is a basis of \mathbb{R}^n consisting of eigenvectors of A .

The Diagonalization Theorem

- 1 An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors. In other words, A is diagonalizable iff there is an eigenvector basis for A .
- 2 $A = PDP^{-1}$ for some diagonal matrix D iff the columns of P are an eigenvector basis. In this case, the diagonal entries of D are the corresponding eigenvalues of A .

Example 1

Is

$$B = \begin{bmatrix} 3 & 10 & 8 & 3 \\ 0 & 4 & 7 & 5 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

diagonalizable?

Solution:

Example 2

Suppose

$$A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}.$$

If the eigenvalues of A are 2 and 1, find matrices P and D with D diagonal such that $A = PDP^{-1}$.

Solution:

Example 2 (cont.)

Example 2 (cont.)

Check your answer!

Next time

Next time:

- Markov chains
- Applications (Google PageRank).
- Difference equations.