MAT 1302B – Mathematical Methods II

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Eigenvectors, eigenvalues, and eigenspaces

Definition (eigenvectors and eigenvalues)

Suppose A is a square matrix. If $\vec{x}$ is a nonzero vector and λ is a scalar such that

$$A\vec{x} = \lambda\vec{x}, \qquad \vec{x} \neq \vec{0},$$

then

- λ is an eigenvalue of A, and
- $\vec{x}$ is an eigenvector of A (an eigenvector corresponding to the eigenvalue λ).

If λ is an eigenvalue, then the set of solutions to $A\vec{x} = \lambda\vec{x}$ (or $(A - \lambda I)\vec{x} = \vec{0}$) is the eigenspace corresponding to λ.

Note: Since the eigenspaces are null spaces of $A - \lambda I$, they are subspaces.

Example 1

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Is $\vec{u}$ or $\vec{v}$ an eigenvector of A?

Solution:

$$A\vec{u} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4\vec{u}$$

Therefore $\vec{u}$ is an eigenvector of A with eigenvalue 4.

$$A\vec{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Since $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is not a multiple of $\vec{v}$, the vector $\vec{v}$ is not an eigenvector of A.

Example 2

Is $\lambda = -3$ an eigenvalue of

$$A = \begin{bmatrix} -4 & -2 & -3 \\ 1 & -1 & 3 \\ -2 & -4 & -9 \end{bmatrix} \quad ?$$

If so, find a basis for the corresponding eigenspace.

Solution:

$$\left[\ A + 3I \ \middle|\ \vec{0}\ \right] = \left[\begin{array}{ccc|c} -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \\ -2 & -4 & -6 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are nontrivial solutions, -3 is an eigenvalue of A. The eigenspace is the solution set.

Example 2 (cont.)

$$\left[\, A+3I \;\middle|\; \vec{0} \,\right] = \left[\begin{array}{ccc|c} -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \\ -2 & -4 & -6 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We write the general solution in vector parametric form:

$$\begin{array}{ll} x_1 & = -2x_2 - 3x_3 \\ x_2 & \text{free} \\ x_3 & \text{free} \end{array} \implies \vec{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

with x_2, x_3 any scalars. Therefore, the eigenspace is

$$\operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and a basis is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Geometric interpretation of eigenvectors/eigenvalues

Mathematica demonstration

http://demonstrations.wolfram.com/EigenvectorsIn2D/

Finding eigenvalues

So we are interested in nontrivial solutions to

$$A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0}.$$

Remember: $M\vec{x} = \vec{0}$ has a nontrivial solution iff M is not invertible and this is true iff $\det M = 0$.

Thus, $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution exactly when $\det(A - \lambda I) = 0$.

Therefore, to find the eigenvalues of A, we need to find the values of λ for which $\det(A - \lambda I) = 0$.

Example 1

Find the eigenvalues of

$$A = \begin{bmatrix} 5 & -4 \\ -3 & 3 \end{bmatrix}.$$

Solution:

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -4 \\ -3 & 3-\lambda \end{bmatrix}$$

Thus we solve

$$0 = \det(A - \lambda I) = (5-\lambda)(3-\lambda) - (-4)(-3) = \lambda^2 - 8\lambda + 3$$

So

$$\lambda = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 3 \cdot 1}}{2} = 4 \pm \frac{\sqrt{52}}{2} = 4 \pm \sqrt{13}$$

Therefore, the eigenvalues of A are $4 + \sqrt{13}$ and $4 - \sqrt{13}$.

Example 2

Find the eigenvalues of

$$B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

and the corresponding eigenspaces.

Solution:

$$B - \lambda I = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$$

So we solve

$$0 = \det(B - \lambda I) = (5-\lambda)^2 - 3^2 = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2)$$

Thus, the eigenvalues of B are 2 and 8.

Example 2 (cont.)

For $\lambda = 2$, the eigenspace is the solution set of $(B - 2I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{c|c} B-2I & \vec{0} \end{array}\right] = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ll} x_1 & = -x_2 \\ x_2 & \text{free} \end{array} \implies \vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad x_2 \text{ any scalar.}$$

So the eigenspace is Span $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 8$, the eigenspace is the solution set of $(B - 8I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{c|c} B-8I & \vec{0} \end{array}\right] = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ll} x_1 & = x_2 \\ x_2 & \text{free} \end{array} \implies \vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_2 \text{ any scalar.}$$

So the eigenspace is Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Example 2 (cont.)

Check your answer!

$$B\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}5 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-5+3\\-3+5\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix} = 2\begin{bmatrix}-1\\1\end{bmatrix} \quad \checkmark$$

$$B\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5+3\\3+5\end{bmatrix} = \begin{bmatrix}8\\8\end{bmatrix} = 8\begin{bmatrix}1\\1\end{bmatrix} \quad \checkmark$$

Example 3

Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the corresponding eigenspaces.

Solution: We solve

$$0 = \det(C - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = (\lambda + i)(\lambda - i).$$

So $\lambda = \pm i$.

We also could have use the quadratic equation:

$$\lambda = \frac{0 \pm \sqrt{0^2 - 4}}{2} = \frac{\pm\sqrt{-4}}{2} = \pm\sqrt{-1} = \pm i.$$

So the eigenvalues are i and $-i$.

Example 3 (cont.)

For $\lambda = i$,

$$\left[\begin{array}{c|c} C - iI & \vec{0} \end{array}\right] = \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{iR_1+R_2} \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{iR_1} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So solution set is

$$\begin{aligned} x_1 &= -ix_2 \\ x_2 &\quad \text{free} \end{aligned} \implies \vec{x} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad x_2 \text{ any scalar.}$$

So the eigenspace is Span $\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

Example 3 (cont.)

For $\lambda = -i$,

$$\left[\begin{array}{c|c} C + iI & \vec{0} \end{array}\right] = \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{-iR_1+R_2} \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-iR_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So solution set is

$$\begin{array}{ll} x_1 & = ix_2 \\ x_2 & \text{free} \end{array} \implies \vec{x} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad x_2 \text{ any scalar.}$$

So the eigenspace is Span $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$.

Example 3 (cont.)

Check your answer!

$$C \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \checkmark$$

$$C \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = -i \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \checkmark$$

The characteristic equation

The determinant transforms the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}$$

into the scalar equation

$$\det(A - \lambda I) = 0.$$

Definition (Characteristic equation and characteristic polynomial)

The equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A.

$\det(A - \lambda I)$ will always be a polynomial in λ (of degree n if A is $n \times n$). It is called the characteristic polynomial of A.

Eigenvalues and transposes

Question: How are the eigenvalues of A and A^T related?

$$\begin{aligned}\det(A-\lambda I) &= \det(A-\lambda I)^T \\ &= det(A^T-(\lambda I)^T) \\ &= \det(A^T-\lambda I^T) \\ &= \det(A^T-\lambda I)\end{aligned}$$

Thus

$$\det(A-\lambda I)=0 \iff \det(A^T-\lambda I)=0$$

Theorem

If A is a square matrix, then A and A^T have the same eigenvalues.

Note: A and A^T can have different eigenvectors!

Example

Find the characteristic polynomial and the eigenvalues of

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ -10 & 7 & 0 & 0 \\ 3 & 6 & 5 & 3 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 0 & 0 & 0 \\ 8 & 3-\lambda & 0 & 0 \\ -10 & 7 & -\lambda & 0 \\ 3 & 6 & 5 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)(-\lambda)(3-\lambda)$$

So the eigenvalues are $-1, 3, 0$ (the entries on the main diagonal).

Notes:

1. We claimed last time that the eigenvalues of triangular matrices are the entries on the diagonal. Now we see why.
2. We say the eigenvalue 3 has multiplicity 2.

Theorem

A scalar λ is an eigenvalue of a square matrix A if and only if it is a solution of the characteristic equation $\det(A - \lambda I) = 0$.

Note: Since $\det(-M) = (-1)^n \det M$ if M is $n \times n$, we have that

$$\det(A - \lambda I) = 0 \iff \det(\lambda I - A) = 0.$$

Therefore, we could also solve $\det(\lambda I - A) = 0$.

Definition (multiplicity)

The (algebraic) multiplicity of an eigenvalue a is the number of times $(\lambda - a)$ (or some multiple of it) appears in the characteristic polynomial (after it is completely factored).

Example

Find the characteristic polynomial of

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 1 & 4 \end{bmatrix}.$$

Solution:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 0 & 1 \\ -2 & 2-\lambda & -1 \\ 0 & 1 & 4-\lambda \end{vmatrix} \\ &= (3-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} + \begin{vmatrix} -2 & 2-\lambda \\ 0 & 1 \end{vmatrix} \\ &= (3-\lambda)((2-\lambda)(4-\lambda)+1) + (-2)(1) \\ &= -\lambda^3 + 9\lambda^2 - 27\lambda + 25 \end{aligned}$$

Multiplicity of eigenvalues

Remarks

1. If A is $n \times n$, the characteristic polynomial of A has degree n (in other words, highest power of λ appearing is n).
2. If A is $n \times n$, it has n eigenvalues, counting multiplicity.
3. Eigenvalues (and entries in eigenvectors) can be complex, even if the entries in the matrix are all real.

Example 1

Find the eigenvalues and their multiplicities of

$$A = \begin{bmatrix} 5 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & -2 \end{bmatrix}.$$

Solution:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 5 & 0 & -6 \\ 2 & \lambda - 1 & -7 \\ -3 & 0 & \lambda + 2 \end{vmatrix} \\ &= (\lambda - 1) \begin{vmatrix} \lambda - 5 & -6 \\ -3 & \lambda + 2 \end{vmatrix} \\ &= (\lambda - 1)((\lambda - 5)(\lambda + 2) - 18) \\ &= (\lambda - 1)(\lambda^2 - 3\lambda - 28) = (\lambda - 1)(\lambda - 7)(\lambda + 4) \end{aligned}$$

Thus the eigenvalues are $1, 7, -4$ and each has multiplicity one.

Example 2

Suppose the characteristic polynomial of a matrix A is

$$\lambda^8 + 6\lambda^7 + 9\lambda^6.$$

What are the eigenvalues of A and their multiplicities?

Solution: We factor the characteristic polynomial:

$$\lambda^8 + 6\lambda^7 + 9\lambda^6 = \lambda^6(\lambda^2 + 6\lambda + 9) = \lambda^6(\lambda + 3)^2$$

Therefore, the eigenvalues are 0 (with multiplicity 6) and -3 (with multiplicity 2).

Eigenvalues and invertibility

Remember: A matrix is invertible iff its determinant is nonzero. Since

$$\det A = \det(A - 0I)$$

we have the following theorem.

Theorem

A matrix is invertible if and only if it does not have zero as an eigenvalue.

Examples

1. If the characteristic polynomial of A is $\lambda^5 + \lambda^3 + \lambda - 2$, is A invertible?

   Solution: Since zero is not a root of the characteristic polynomial, zero is not an eigenvalue of A. Thus, A is invertible.

2. If the characteristic polynomial of B is $\lambda^9 + 6\lambda^4 - 2\lambda$, is B invertible?

   Solution: Zero is a root of the characteristic polynomial of B and hence an eigenvalue of B. Therefore, B is not invertible.

Finding eigenvalues and eigenvectors/eigenspaces

Procedure for finding eigenvalues, eigenvectors and eigenspaces

1. To find the eigenvalues, find the solutions to the characteristic equation
$$\det(A - \lambda I) = 0.$$
2. For each each eigenvalue, solve the equation
$$(A - \lambda I)\vec{x} = \vec{0}$$
to find the corresponding eigenspace.
3. The nonzero vectors in each eigenspace are the eigenvectors corresponding to the given eigenvalue.

Next time

For next time: Read Section SD.

- Diagonalization.
- How to use eigenvectors/eigenvalues to make a matrix diagonal.
- Useful since diagonal matrices are particularly easy to work with.