MAT 1302B – Mathematical Methods II

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Eigenvectors, eigenvalues, and eigenspaces

Definition (eigenvectors and eigenvalues)

Suppose A is a square matrix. If $\vec{x}$ is a nonzero vector and λ is a scalar such that

$$A\vec{x} = \lambda\vec{x}, \qquad \vec{x} \neq \vec{0},$$

then

- λ is an eigenvalue of A, and
- $\vec{x}$ is an eigenvector of A (an eigenvector corresponding to the eigenvalue λ).

If λ is an eigenvalue, then the set of solutions to $A\vec{x} = \lambda\vec{x}$ (or $(A - \lambda I)\vec{x} = \vec{0}$) is the eigenspace corresponding to λ.

Note: Since the eigenspaces are null spaces of $A - \lambda I$, they are subspaces.

Example 1

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Is $\vec{u}$ or $\vec{v}$ an eigenvector of A?

Solution:

Example 2

Is $\lambda = -3$ an eigenvalue of

$$A = \begin{bmatrix} -4 & -2 & -3 \\ 1 & -1 & 3 \\ -2 & -4 & -9 \end{bmatrix} \quad ?$$

If so, find a basis for the corresponding eigenspace.

Solution:

Example 2 (cont.)

Geometric interpretation of eigenvectors/eigenvalues

Mathematica demonstration

http://demonstrations.wolfram.com/EigenvectorsIn2D/

Finding eigenvalues

So we are interested in nontrivial solutions to

$$A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0}.$$

Remember: $M\vec{x} = \vec{0}$ has a nontrivial solution iff M is not invertible and this is true iff $\det M = 0$.

Thus, $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution exactly when $\det(A - \lambda I) = 0$.

Therefore, to find the eigenvalues of A, we need to find the values of λ for which $\det(A - \lambda I) = 0$.

Example 1

Find the eigenvalues of

$$A = \begin{bmatrix} 5 & -4 \\ -3 & 3 \end{bmatrix}.$$

Solution:

Example 2

Find the eigenvalues of

$$B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

and the corresponding eigenspaces.

Solution:

Example 2 (cont.)

Example 2 (cont.)

Check your answer!

Example 3

Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the corresponding eigenspaces.

Solution:

Example 3 (cont.)

Example 3 (cont.)

Example 3 (cont.)

Check your answer!

The characteristic equation

The determinant transforms the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}$$

into the scalar equation

$$\det(A - \lambda I) = 0.$$

Definition (Characteristic equation and characteristic polynomial)

The equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A.

$\det(A - \lambda I)$ will always be a polynomial in λ (of degree n if A is $n \times n$). It is called the characteristic polynomial of A.

Eigenvalues and transposes

Question: How are the eigenvalues of A and A^T related?

$$\begin{aligned}
\det(A - \lambda I) &= \det(A - \lambda I)^T \\
&= det(A^T - (\lambda I)^T) \\
&= \det(A^T - \lambda I^T) \\
&= \det(A^T - \lambda I)
\end{aligned}$$

Thus

$$\det(A - \lambda I) = 0 \iff \det(A^T - \lambda I) = 0$$

Theorem

If A is a square matrix, then A and A^T have the same eigenvalues.

Note: A and A^T can have different eigenvectors!

Example

Find the characteristic polynomial and the eigenvalues of

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ -10 & 7 & 0 & 0 \\ 3 & 6 & 5 & 3 \end{bmatrix}.$$

Solution:

Theorem

A scalar λ is an eigenvalue of a square matrix A if and only if it is a solution of the characteristic equation $\det(A - \lambda I) = 0$.

Note: Since $\det(-M) = (-1)^n \det M$ if M is $n \times n$, we have that

$$\det(A - \lambda I) = 0 \iff \det(\lambda I - A) = 0.$$

Therefore, we could also solve $\det(\lambda I - A) = 0$.

Definition (multiplicity)

The (algebraic) multiplicity of an eigenvalue a is the number of times $(\lambda - a)$ (or some multiple of it) appears in the characteristic polynomial (after it is completely factored).

Example

Find the characteristic polynomial of

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 1 & 4 \end{bmatrix}.$$

Solution:

Multiplicity of eigenvalues

Remarks

1. If A is $n \times n$, the characteristic polynomial of A has degree n (in other words, highest power of λ appearing is n).

2. If A is $n \times n$, it has n eigenvalues, counting multiplicity.

3. Eigenvalues (and entries in eigenvectors) can be complex, even if the entries in the matrix are all real.

Example 1

Find the eigenvalues and their multiplicities of

$$A = \begin{bmatrix} 5 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & -2 \end{bmatrix}.$$

Solution:

Example 2

Suppose the characteristic polynomial of a matrix A is

$$\lambda^8 + 6\lambda^7 + 9\lambda^6.$$

What are the eigenvalues of A and their multiplicities?

Solution:

Eigenvalues and invertibility

Remember: A matrix is invertible iff its determinant is nonzero. Since

$$\det A = \det(A - 0I)$$

we have the following theorem.

Theorem

A matrix is invertible if and only if it does not have zero as an eigenvalue.

Examples

1. If the characteristic polynomial of A is $\lambda^5 + \lambda^3 + \lambda - 2$, is A invertible?

   Solution:

2. If the characteristic polynomial of B is $\lambda^9 + 6\lambda^4 - 2\lambda$, is B invertible?

   Solution:

Finding eigenvalues and eigenvectors/eigenspaces

Procedure for finding eigenvalues, eigenvectors and eigenspaces

1. To find the eigenvalues, find the solutions to the characteristic equation
$$\det(A - \lambda I) = 0.$$
2. For each each eigenvalue, solve the equation
$$(A - \lambda I)\vec{x} = \vec{0}$$
to find the corresponding eigenspace.
3. The nonzero vectors in each eigenspace are the eigenvectors corresponding to the given eigenvalue.

Next time

For next time: Read Section SD.

- Diagonalization.
- How to use eigenvectors/eigenvalues to make a matrix diagonal.
- Useful since diagonal matrices are particularly easy to work with.