MAT 1302B – Mathematical Methods II

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Winter 2015 – Lecture 17

Announcements

Third Midterm Exam

- March 27.
- Covers material up to and including last lecture.
- No calculators.
- Bring your student ID.
- **Know your DGD number, write in pen.**

Last time:

- Complex numbers

Today:

- Eigenvectors/eigenvalues

Eigenvectors and eigenvalues

Definition (eigenvectors and eigenvalues)

Suppose A is a square matrix. If $\vec{x}$ is a nonzero vector and λ is a scalar such that

$$A\vec{x} = \lambda\vec{x}$$

then

- λ is an eigenvalue of A, and
- $\vec{x}$ is an eigenvector of A (an eigenvector corresponding to the eigenvalue λ).

Example

$$A = \begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then

$$A\vec{u} = \begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \vec{u} = 1 \cdot \vec{u}.$$

So $\vec{u}$ is an eigenvector of A with eigenvalue 1.

$$A\vec{v} = \begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -10 \end{bmatrix}$$

If $\vec{v}$ were an eigenvector, we would have

$$\begin{bmatrix} -7 \\ -10 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\lambda \\ -\lambda \end{bmatrix}.$$

But

$$-7 = 2\lambda \implies \lambda = \frac{-7}{2}, \quad -10 = -\lambda \implies \lambda = 10$$

which is a contradiction. So $\vec{v}$ is not an eigenvector of A.

Notes

1. "Eigen" = "own", "characteristic".
    - Emphasizes how important eigenvectors/eigenvalues are for describing a matrix.

2. Eigenvalues (and entries in eigenvectors) can be complex numbers.

3. An eigenvector cannot equal $\vec{0}$ but an eigenvalue can be zero.
    - Example:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix with eigenvalue 0.

Example

Is $\lambda = 2$ an eigenvalue of

$$A = \begin{bmatrix} -7 & -3 \\ -3 & 1 \end{bmatrix} \quad ?$$

If so, find an eigenvector corresponding to this eigenvalue.

Solution: $\lambda = 2$ is an eigenvalue iff $A\vec{x} = 2\vec{x}$ has a nontrivial solution.

$$A\vec{x} = 2\vec{x} \iff A\vec{x} - 2\vec{x} = \vec{0} \iff (A - 2I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{c|c} A - 2I & \vec{0} \end{array}\right] = \left[\begin{array}{cc|c} -9 & -3 & 0 \\ -3 & -1 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

- This has a nontrivial solution.
- So 2 is an eigenvalue of A.
- Any nontrivial solution to the system gives an eigenvector.

Example (cont.)

$$\left[\ A - 2I \ \middle|\ \vec{0}\ \right] = \left[\begin{array}{cc|c} -9 & -3 & 0 \\ -3 & -1 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$x_1 = \frac{-1}{3} x_2,$$
$$x_2 \quad \text{free.}$$

Take $x_2 = 3$, then

$$\vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \text{is an eigenvector.}$$

Note: Since any nonzero solution gives an eigenvector, the set of all eigenvectors is the set of nonzero vectors in

$$\left\{ \begin{bmatrix} \frac{-1}{3} x_2 \\ x_2 \end{bmatrix} \middle|\ x_2 \text{ any scalar} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} \frac{-1}{3} \\ 1 \end{bmatrix} \right\}.$$

Example (cont.)

Check your answer!!

We concluded that

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

is an eigenvector of A with eigenvalue 2.

So we multiply

$$A \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}. \qquad \checkmark$$

Procedure for finding eigenvectors

1. To determine whether or not λ is an eigenvalue for A, use row reduction to determine if

$$(A - \lambda I)\vec{x} = \vec{0}$$

has a nontrivial solution.

2. To find the eigenvectors corresponding to an eigenvalue λ, find the nontrivial solutions to

$$(A - \lambda I)\vec{x} = \vec{0}.$$

Definition (eigenspace)

If λ is an eigenvalue of A, then the set of all solutions to $A\vec{x} = \lambda\vec{x}$ (equivalently, to $(A - \lambda I)\vec{x} = \vec{0}$) is called the eigenspace of A corresponding to λ.

Since the eigenspace is simply the null space of the matrix $A - \lambda I$, it is a subspace of $\mathbb{R}^n$ (if A is $n \times n$).

Example 1

Is $\lambda = 3$ an eigenvalue of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad ?$$

If so, find the corresponding eigenspace and a basis for this eigenspace.

Solution:

$$\left[\ A - 3I \ \middle|\ \vec{0}\ \right] = \left[\begin{array}{ccc|c} -2 & 3 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This has nontrivial solutions, so YES, 3 is an eigenvalue of A.

Example 1 (cont.)

The eigenspace is given by the set of solutions:

$$x_1 = \frac{3}{4}x_3$$
$$x_2 = \frac{1}{2}x_3$$
$$x_3 \quad \text{free}$$

So the eigenspace is

$$\left\{ x_3 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \middle| \; x_3 \text{ any scalar} \right\} = \text{Span} \left\{ \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}.$$

Example 2

Is $\lambda = 3$ an eigenvalue of

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 2 & 1 & 4 \\ 3 & -3 & 9 \end{bmatrix} \quad ?$$

If so, find the corresponding eigenspace and a basis for this eigenspace.

Solution:

$$\left[\ A - 3I \ \middle|\ \vec{0}\ \right] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 6 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This has nontrivial solutions, so YES, 3 is an eigenvalue of A.

Example 2 (cont.)

The eigenspace is given by the set of solutions:

$$\begin{aligned} x_1 &= x_2 - 2x_3 \\ x_2 &\quad \text{free} \\ x_3 &\quad \text{free} \end{aligned}$$

So the eigenspace is

$$\left\{ \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_2, x_3 \text{ scalars} \right\} = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \middle| \, x_2, x_3 \text{ scalars} \right\}$$
$$= \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Example 3

If

$$A = \begin{bmatrix} 0 & 4 & -1 \\ 2 & 5 & 0 \\ -4 & -5 & 0 \end{bmatrix},$$

is $\lambda = -2$ an eigenvalue of A?

Solution: We solve $(A - (-2)I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{c|c} A + 2I & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 2 & 7 & 0 & 0 \\ -4 & -5 & 2 & 0 \end{array}\right] \xrightarrow[2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since there are no free variables, the system has no nontrivial solutions. Therefore the answer is NO, -2 is not an eigenvalue of A.

Eigenvalues of triangular matrices

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal. (We'll see why later.)

Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 7 & 9 & -1 \end{bmatrix} \qquad \text{Eigenvalues: } 2, 3, -1$$

Example 2

$$B = \begin{bmatrix} 2 & 7 & 10 & 9 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad \text{Eigenvalues: } 2, 0, -5$$

Eigenvalues and powers of matrices

Question: If we know the eigenvalues of A, what can we say about the eigenvalues of A^n for $n \geq 1$?

Note that if $\vec{x}$ is an eigenvector of A corresponding to eigenvalue λ, then

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A^2\vec{x} &= A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x} \\ A^3\vec{x} &= A(A^2\vec{x}) = A(\lambda^2\vec{x}) = \lambda^2(A\vec{x}) = \lambda^2(\lambda\vec{x}) = \lambda^3\vec{x} \\ &\vdots \\ A^n\vec{x} &= \lambda^n\vec{x} \end{aligned}$$

Proposition

If the eigenvalues of A are $\lambda_1, \ldots, \lambda_k$, then the eigenvalues of A^n are $\lambda_1^n, \ldots, \lambda_k^n$.

Example

If

$$A = \begin{bmatrix} 2 & 7 & 8 & 9 & -5 \\ 0 & 0 & 3 & -5 & 6 \\ 0 & 0 & -2 & 8 & 10 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

then what are the eigenvalues of A^3 and A^4?

Solution:

- Since A is triangular, its eigenvalues are the entries on the diagonal.
- So the eigenvalues of A are $0, -1, 1, -2, 2$.
- The eigenvalues of A^3 are $0, -1, 1, -8, 8$.
- The eigenvalues of A^4 are $0, 1, 16$.

Eigenvectors with distinct eigenvalues

Theorem

If $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of a matrix A, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Why?

- Suppose that this is false and $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are dependent.
- Then one of the vectors is a linearly combination of the previous ones. Let $\vec{v}_{p+1}$ be the first vector with this property:

$$c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p = \vec{v}_{p+1} \tag{1}$$

Multiply both sides by A:

$$c_1 A\vec{v}_1 + \cdots + c_p A\vec{v}_p = A\vec{v}_{p+1}$$

$$c_1 \lambda_1 \vec{v}_1 + \cdots + c_p \lambda_p \vec{v}_p = \lambda_{p+1} \vec{v}_{p+1} \tag{2}$$

Multiply (1) by λ_{p+1} and subtract from (2).

Eigenvectors with distinct eigenvalues

Why? (cont.)

$$c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p = \vec{0}$$

- Since $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are linearly independent (by our choice of $\vec{v}_{p+1}$), the above coefficients are all zero.
- Since all the eigenvalues are distinct, none of the $\lambda_i - \lambda_{p+1}$ are zero.
- Thus, all the c_i are zero.
- Then

$$\vec{v}_{p+1} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}.$$

- This is a contradiction because eigenvectors can't be zero.
- Therefore, the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Importance of eigenvectors

1. Mathematics (theory)
    - Diagonalization.
    - Linear transformations.
2. Applications
    - Economics, social science: Markov chains.
    - Music/sounds (harmony, noise dampening).
    - Internet search engines (Google PageRank).
3. Physics
    - Quantum mechanics.
    - Heisenberg uncertainty principle.

... and many more!

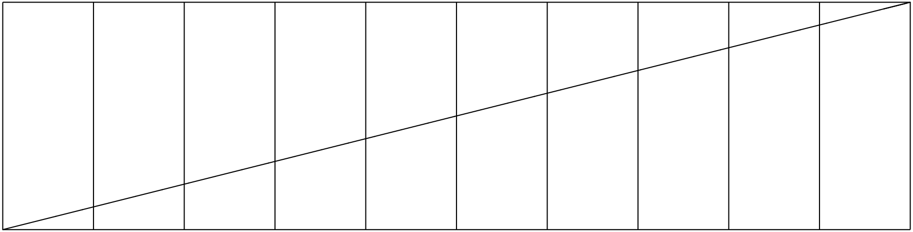
Weekend problem – last time

Question: Why do bills (20 dollar bill, etc.) have two serial numbers?

More precisely, if they didn't, how could you easily cut up ten 20 dollar bills and tape them together in such a way that you ended up with eleven bills that you could probably get away with spending?

Hint: Each of the new bills will be slightly smaller than a real bill.

Weekend problem – solution



- Place bills side-by-side.
- Cut diagonally from corner to corner.
- Slide the bottom halves to the right by one bill.
- Tape the pieces together.
- You end up with one extra bill, each one slightly shorter than a real bill.
- However, the serial numbers won't match!

Next time

For next time: Read Section EE, PEE.

- Today we learned how to check if some given scalar is an eigenvalue.
- Next time, we learn how to find eigenvalues.
- Related to determinants.