MAT 1302B – Mathematical Methods II

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Announcements

Third Midterm Exam

- March 27.
- Covers material up to and including last lecture.
- No calculators.
- Bring your student ID.
- **Know your DGD number, write in pen.**

Last time:

- Complex numbers

Today:

- Eigenvectors/eigenvalues

Eigenvectors and eigenvalues

Definition (eigenvectors and eigenvalues)

Suppose A is a square matrix. If $\vec{x}$ is a nonzero vector and λ is a scalar such that

$$A\vec{x} = \lambda\vec{x}$$

then

- λ is an eigenvalue of A, and
- $\vec{x}$ is an eigenvector of A (an eigenvector corresponding to the eigenvalue λ).

Example

$$A = \begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then

$$A\vec{u} = \begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} =$$

$$A\vec{v} = \begin{bmatrix} -3 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} =$$

If $\vec{v}$ were an eigenvector, we would have

Notes

1. "Eigen" = "own", "characteristic".
   - Emphasizes how important eigenvectors/eigenvalues are for describing a matrix.

2. Eigenvalues (and entries in eigenvectors) can be complex numbers.

3. An eigenvector cannot equal $\vec{0}$ but an eigenvalue can be zero.
   - Example:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example

Is $\lambda = 2$ an eigenvalue of

$$A = \begin{bmatrix} -7 & -3 \\ -3 & 1 \end{bmatrix} \quad ?$$

If so, find an eigenvector corresponding to this eigenvalue.

Solution:

Example (cont.)

Example (cont.)

Check your answer!!

Procedure for finding eigenvectors

1. To determine whether or not λ is an eigenvalue for A, use row reduction to determine if

$$(A - \lambda I)\vec{x} = \vec{0}$$

has a nontrivial solution.

2. To find the eigenvectors corresponding to an eigenvalue λ, find the nontrivial solutions to

$$(A - \lambda I)\vec{x} = \vec{0}.$$

Definition (eigenspace)

If λ is an eigenvalue of A, then the set of all solutions to $A\vec{x} = \lambda\vec{x}$ (equivalently, to $(A - \lambda I)\vec{x} = \vec{0}$) is called the eigenspace of A corresponding to λ.

Since the eigenspace is simply the null space of the matrix $A - \lambda I$, it is a subspace of $\mathbb{R}^n$ (if A is $n \times n$).

Example 1

Is $\lambda = 3$ an eigenvalue of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad ?$$

If so, find the corresponding eigenspace and a basis for this eigenspace.

Solution:

Example 1 (cont.)

Example 2

Is $\lambda = 3$ an eigenvalue of

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 2 & 1 & 4 \\ 3 & -3 & 9 \end{bmatrix} \quad ?$$

If so, find the corresponding eigenspace and a basis for this eigenspace.

Solution:

Example 2 (cont.)

Example 3

If

$$A = \begin{bmatrix} 0 & 4 & -1 \\ 2 & 5 & 0 \\ -4 & -5 & 0 \end{bmatrix},$$

is $\lambda = -2$ an eigenvalue of A?

Solution:

Eigenvalues of triangular matrices

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal. (We'll see why later.)

Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 7 & 9 & -1 \end{bmatrix} \qquad \text{Eigenvalues:}$$

Example 2

$$B = \begin{bmatrix} 2 & 7 & 10 & 9 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad \text{Eigenvalues:}$$

Eigenvalues and powers of matrices

Question: If we know the eigenvalues of A, what can we say about the eigenvalues of A^n for $n \geq 1$?

Note that if $\vec{x}$ is an eigenvector of A corresponding to eigenvalue λ, then

$$
\begin{aligned}
A\vec{x} &= \lambda\vec{x} \\
A^2\vec{x} &= A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x} \\
A^3\vec{x} &= A(A^2\vec{x}) = A(\lambda^2\vec{x}) = \lambda^2(A\vec{x}) = \lambda^2(\lambda\vec{x}) = \lambda^3\vec{x} \\
&\vdots \\
A^n\vec{x} &= \lambda^n\vec{x}
\end{aligned}
$$

Proposition

If the eigenvalues of A are $\lambda_1, \ldots, \lambda_k$, then the eigenvalues of A^n are $\lambda_1^n, \ldots, \lambda_k^n$.

Example

If

$$A = \begin{bmatrix} 2 & 7 & 8 & 9 & -5 \\ 0 & 0 & 3 & -5 & 6 \\ 0 & 0 & -2 & 8 & 10 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

then what are the eigenvalues of A^3 and A^4?

Solution:

Eigenvectors with distinct eigenvalues

Theorem

If $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of a matrix A, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Why?

- Suppose that this is false and $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are dependent.
- Then one of the vectors is a linearly combination of the previous ones. Let $\vec{v}_{p+1}$ be the first vector with this property:

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{v}_{p+1} \tag{1}$$

Multiply both sides by A:

$$c_1 A\vec{v}_1 + \cdots + c_p A\vec{v}_p = A\vec{v}_{p+1}$$
$$c_1\lambda_1\vec{v}_1 + \cdots + c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1} \tag{2}$$

Multiply (1) by λ_{p+1} and subtract from (2).

Eigenvectors with distinct eigenvalues

Why? (cont.)

$$c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p = \vec{0}$$

- Since $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are linearly independent (by our choice of $\vec{v}_{p+1}$), the above coefficients are all zero.
- Since all the eigenvalues are distinct, none of the $\lambda_i - \lambda_{p+1}$ are zero.
- Thus, all the c_i are zero.
- Then

$$\vec{v}_{p+1} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}.$$

- This is a contradiction because eigenvectors can't be zero.
- Therefore, the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Importance of eigenvectors

1. Mathematics (theory)
   - Diagonalization.
   - Linear transformations.
2. Applications
   - Economics, social science: Markov chains.
   - Music/sounds (harmony, noise dampening).
   - Internet search engines (Google PageRank).
3. Physics
   - Quantum mechanics.
   - Heisenberg uncertainty principle.

... and many more!

Weekend problem – last time

Question: Why do bills (20 dollar bill, etc.) have two serial numbers?

More precisely, if they didn't, how could you easily cut up ten 20 dollar bills and tape them together in such a way that you ended up with eleven bills that you could probably get away with spending?

Hint: Each of the new bills will be slightly smaller than a real bill.

Weekend problem – solution

Next time

For next time: Read Section EE, PEE.

- Today we learned how to check if some given scalar is an eigenvalue.
- Next time, we learn how to find eigenvalues.
- Related to determinants.