MAT 1302B – Mathematical Methods II

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Winter 2015 – Lecture 16

Announcements

Third Midterm Exam:

- Friday, March 27.
- Covers material up to and including Lecture 16.

Course Evaluations:

- In class on Tuesday (March 24).
- Please come to class and be on time.
- Your feedback is important.

Today:

- Complex numbers.

Polynomials and roots

Definition (polynomial)

A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, \ldots, a_n \in \mathbb{R}$ and n is some non-negative integer.

Examples

1. $3x^5 - 4x^4 + 2x^2 - 8$
2. $7x^9 + 5x$

Definition (roots of polynomials)

If $f(x)$ is a polynomial, then a root of $f(x)$ is a number r such that $f(r) = 0$.

Examples

1. If $c \in \mathbb{R}$, then $f(x) = x - c$ is a polynomial. The only root of $f(x)$ is c:
$$f(c) = c - c = 0.$$

2. If $c \in \mathbb{R}$, then $f(x) = x^2 - c$ is a polynomial. Then r is a root if $f(r) = r^2 - c = 0$, i.e. if $r^2 = c$.
    - If $c > 0$, then $f(x)$ has two roots: $\pm\sqrt{c}$.
    - If $c = 0$, then $f(x)$ has one root: 0.
    - If $c < 0$, then $f(x)$ has no roots.

So if we only consider real numbers, then some polynomials have roots and some do not.

Complex numbers will fix this discrepancy.

Complex numbers

Definition (complex number)

A complex number is a number written in the form

$$z = a + bi$$

where

- $a \in \mathbb{R}$ is the real part of z ($\operatorname{Re} z$),
- $b \in \mathbb{R}$ is the imaginary part of z ($\operatorname{Im} z$),
- i is a formal symbol satisfying $i^2 = -1$.

Example

If $z = 3 + (-7)i$, then

$$\operatorname{Re} z = 3, \qquad \operatorname{Im} z = -7.$$

We write $z = 3 - 7i$.

Complex numbers

Real numbers

Real numbers are just complex numbers with zero imaginary part:

$$a + 0i, \qquad a \in \mathbb{R}.$$

Imaginary numbers

Complex numbers with zero real part are called imaginary numbers:

$$bi = 0 + bi, \qquad b \in \mathbb{R}.$$

Notation

We denote the set of complex numbers by $\mathbb{C}$.

So $z \in \mathbb{C}$ means that z is a complex number.

Arithmetic with complex numbers

Addition of complex numbers

We add complex numbers by adding their real and imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \qquad a, b, c, d \in \mathbb{R}.$$

Multiplication of complex numbers

We multiply complex numbers by expanding and using $i^2 = -1$.

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Arithmetic with complex numbers

Examples

1. $(2-3i)+(5+i)=7-2i$
2. $(2-3i)(5+i)=10+2i-15i-3i^2=10-13i-3(-1)=13-13i$
3. $4(-4+5i)=-16+20i$
4. Q: Which complex numbers z satisfy $z^2=-4$?
   A: $2i, -2i$.
5. Powers of i:
   - $i^0=1$
   - $i^1=i$
   - $i^2=-1$
   - $i^3=-i$
   - $i^4=1$
   - $i^5=i$ (the cycle repeats...)

Arithmetic with complex numbers

Subtraction of complex numbers

We define subtraction by

$$z - w = z + (-1)w, \qquad z, w \in \mathbb{C}.$$

Example

$$(2 - 3i) - (4 - 7i) = -2 + 4i$$

Complex conjugation

Definition (complex conjugation)

The conjugate of $z = a + bi$ $(a, b \in \mathbb{R})$ is

$$\bar{z} = a - bi$$

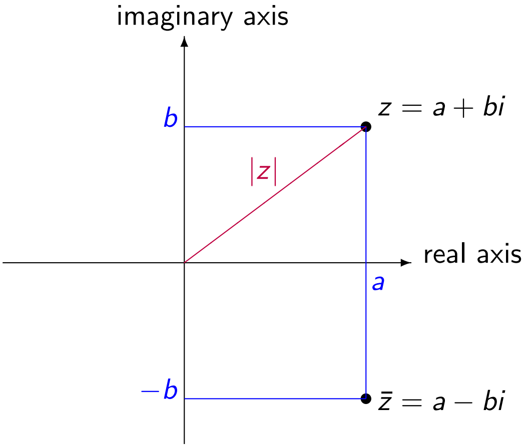
(we negate the imaginary part).

Examples

1. $\overline{-5 + 2i} = -5 - 2i$
2. $\overline{3 - \frac{2}{5}i} = 3 + \frac{2}{5}i$

Geometric interpretation

Every complex number $a + bi$ corresponds to a point (a, b) (or vector) in the plane $\mathbb{R}^2$.



$|z| = \sqrt{a^2 + b^2}$ is the distance of z from the origin (Pythagorean Theorem)

Geometric interpretation (cont.)

Complex number demonstration

http://demonstrations.wolfram.com/ComplexNumber/

Absolute values

If $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2 \geq 0.$$

Definition (absolute value or modulus)

If $z \in \mathbb{C}$, then the absolute value (or modulus) of z is

$$|z| \stackrel{\text{def}}{=} \sqrt{z\bar{z}}.$$

So if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}.$$

$|z|$ is always a nonnegative number (the distance from the origin to z, viewed as a point in the plane).

Note

If z is a real number, then $z = a + 0i$ and so

$$|z| = \sqrt{a^2}$$

which is the usual absolute value of a real number.

So our new definition agrees with the usual one for the special case of real numbers.

Examples

1. If $z = -4 + 5i$, then

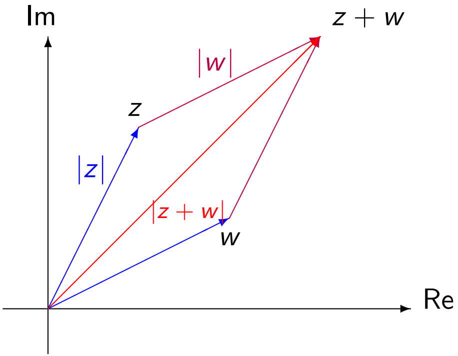
$$|z| = \sqrt{(-4)^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}.$$

2. If $z = 3 + 4i$, then

$$|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Geometric interpretation (cont.)

Addition of complex numbers corresponds to vector addition.



Recall: The length of one side of a triangle is less than the sum of lengths of the other two sides (the shortest distance between two points is a straight line). Thus

$$|z + w| \leq |z| + |w|.$$

This is called the triangle inequality.

Geometric interpretation (cont.)

Demonstration of addition of complex numbers

http://demonstrations.wolfram.com/ComplexAddition/

Some properties of complex numbers

If z and w are complex numbers, then

1. $\bar{z} = z$ iff $z \in \mathbb{R}$,
2. $\overline{w + z} = \bar{w} + \bar{z}$,
3. $\overline{wz} = \bar{w}\bar{z}$ (so, if $r \in \mathbb{R}$, then $\overline{rz} = \bar{r}\bar{z} = r\bar{z}$),
4. $z\bar{z} = |z|^2 \geq 0$,
5. $|wz| = |w||z|$,
6. triangle inequality: $|w + z| \leq |w| + |z|$.

Multiplicative inverses and division

We've defined addition, subtraction and multiplication of complex numbers.

What about division?

Division is related to multiplicative inverses. For instance, if $a, b \in \mathbb{R}$, $b \neq 0$, then

$$a \div b = \frac{a}{b} = a \cdot \frac{1}{b} = a \cdot b^{-1}.$$

Dividing a complex number by a real number is easy: If $a, b, c \in \mathbb{R}$, $c \neq 0$, and $z = a + bi$, then

$$\frac{z}{c} = \frac{a + bi}{c} = \frac{1}{c}(a + bi) = \frac{a}{c} + \frac{b}{c}i.$$

Multiplicative inverses and division (cont.)

If $z \neq 0$, then $|z| > 0$ and

$$z \cdot \frac{\bar{z}}{|z|^2} = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

and so we write

$$\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2}.$$

This is the multiplicative inverse of z.

Example

If $z = 3 - 4i$, then

$$\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{3+4i}{3^2+4^2} = \frac{3+4i}{25} = \frac{3}{25} + \frac{4}{25}i.$$

Multiplicative inverses and division (cont.)

Definition (quotients)

If $z, w \in \mathbb{C}$ and $w \neq 0$, then we define

$$\frac{z}{w} = z \cdot \frac{1}{w}.$$

Example

Suppose $w = 2 - 3i$ and $z = 5 + i$. Then

$$\begin{aligned} \frac{w}{z} &= w \cdot \frac{1}{z} = w \cdot \frac{\bar{z}}{|z|^2} \\ &= (2-3i) \cdot \frac{5-i}{5^2+1^2} = \frac{10 - 2i - 15i - 3}{26} \\ &= \frac{7-17i}{26} = \frac{7}{26} - \frac{17}{26} i. \end{aligned}$$

Geometric interpretation of multiplication of complex numbers

Demonstration of multiplication of complex numbers

http://demonstrations.wolfram.com/ComplexMultiplication/

Example

Simplify

$$\frac{3+2i}{-4-3i}.$$

That is, write it in the form $a+bi$, $a, b \in \mathbb{R}$.

Solution:

$$\frac{3+2i}{-4-3i} = \frac{3+2i}{-4-3i} \cdot \frac{-4+3i}{-4+3i}$$

(mult numerator and denominator by conjugate of denominator)

$$= \frac{-12+9i-8i-6}{16+9}$$

$$= \frac{-18+i}{25} = -\frac{18}{25} + \frac{1}{25}i$$

Complex numbers in linear algebra

- For everything we've done so far, we can use complex numbers as the scalars instead of real numbers.

  For example:
  - linear equations/systems can involve complex numbers,
  - matrices can have complex numbers as entries,
  - we have discuss vectors in $\mathbb{C}^n$ – these are just $n \times 1$ matrices with complex number entries,
  - we can talk about subspaces of $\mathbb{C}^n$ instead of $\mathbb{R}^n$.

- We will see that certain statements/theorems are only true if we work with complex numbers.

Examples

1.
$$\begin{bmatrix} 3 & i \\ 0 & -i \end{bmatrix} \begin{bmatrix} -1 & 2i \\ 0 & 5i \end{bmatrix} = \begin{bmatrix} -3 & -5+6i \\ 0 & 5 \end{bmatrix}$$

2.
$$\text{Span}\left\{ \begin{bmatrix} 2i \\ 3+i \\ -4i \end{bmatrix}, \begin{bmatrix} 4 \\ -2-i \\ 4+7i \end{bmatrix} \right\} \text{ is a subspace of } \mathbb{C}^3.$$

3. Row reduction involving complex numbers:

$$\begin{bmatrix} 1 & i & -2 \\ -i & 1 & 3+2i \\ 2i & -2 & 6-4i \end{bmatrix} \xrightarrow[-2iR_1+R_3]{iR_1+R_2} \begin{bmatrix} 1 & i & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & i & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & i & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_2+R_1} \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{RREF}$$

Example

Solve the following equation for z:

$$(z + 3 - 2i)(1 + i) = 5$$

Solution: We first multiply both sides by $(1 + i)^{-1}$:

$$z + 3 - 2i = 5\,(1 + i)^{-1} = 5\,\frac{1 - i}{1^2 + 1^2} = \frac{5}{2} - \frac{5}{2}i$$

Then we add $-3 + 2i$ to both sides of the equation:

$$z = \left(\frac{5}{2} - \frac{5}{2}i\right) + (-3 + 2i) = -\frac{1}{2} - \frac{1}{2}i.$$

Fundamental Theorem of Algebra

One of the most important properties of the complex numbers is that there are beautiful mathematical theorems that are only true if you work with complex numbers.

Fundamental Theorem of Algebra

Every non-constant single-variable polynomial with complex coefficients has at least one complex root.

Remember, that this was not true if we only considered real numbers.

Example

- Consider the polynomial $f(x) = x^2 + 1$.
- It has no real roots since if $r \in \mathbb{R}$, then $f(r) = r^2 + 1 > 0$.
- However, it has two complex roots ($\pm i$) since

$$f(i) = i^2 + 1 = -1 + 1 = 0, \quad f(-i) = (-i)^2 + 1 = -1 + 1 = 0.$$

Complex numbers and electronics (aside)

Basic electric circuits contain voltage sources (battery or power outlet), resistors, capacitors, and inductors.

If your voltage source is alternating (like the voltage in your home), then

- the effect of a resistor depends on the current running through it, and
- the effect of an inductor depends on the rate of change of the current running through it.

The equations to describe a circuit can get quite complicated. However, if we use complex numbers, then the roll of resistors and inductors is basically the same except:

- resistors have "impedances" that are real numbers, and
- inductors have "impedances" that are imaginary numbers!

Moral: Working with complex numbers greatly simplifies the problem.

Weekend problem

Question: Why do bills (20 dollar bill, etc.) have two serial numbers?

More precisely, if they didn't, how could you easily cut up ten 20 dollar bills and tape them together in such a way that you ended up with eleven bills that you could probably get away with spending?

Hint: Each of the new bills will be slightly smaller than a real bill.

Next time

For next time: Read Sections EE, PEE.

- Eigenvectors.
- Eigenvalues.
- These are topics where complex numbers are essential.