

MAT 1302B – Mathematical Methods II

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Announcements

Third Midterm Exam:

- Friday, March 27.
- Covers material up to and including Lecture 16.

Course Evaluations:

- In class on Tuesday (March 24).
- Please come to class and be on time.
- Your feedback is important.

Today:

- Complex numbers.

Polynomials and roots

Definition (polynomial)

A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, \dots, a_n \in \mathbb{R}$ and n is some non-negative integer.

Examples

① $3x^5 - 4x^4 + 2x^2 - 8$

② $7x^9 + 5x$

Definition (roots of polynomials)

If $f(x)$ is a polynomial, then a **root** of $f(x)$ is a number r such that $f(r) = 0$.

Examples

- 1 If $c \in \mathbb{R}$, then $f(x) = x - c$ is a polynomial. The only root of $f(x)$ is
- 2 If $c \in \mathbb{R}$, then $f(x) = x^2 - c$ is a polynomial. Then r is a root if $f(r) = r^2 - c = 0$, i.e. if $r^2 = c$.



Complex numbers

Definition (complex number)

A **complex number** is a number written in the form

$$z = a + bi$$

where

- $a \in \mathbb{R}$ is the **real part** of z ($\operatorname{Re} z$),
- $b \in \mathbb{R}$ is the **imaginary part** of z ($\operatorname{Im} z$),
- i is a formal symbol satisfying $i^2 = -1$.

Example

If $z = 3 + (-7)i$, then

$\operatorname{Re} z =$

$\operatorname{Im} z =$

We write $z = 3 - 7i$.

Complex numbers

Real numbers

Real numbers are just complex numbers with zero imaginary part:

$$a + 0i, \quad a \in \mathbb{R}.$$

Imaginary numbers

Complex numbers with zero real part are called **imaginary numbers**:

$$bi = 0 + bi, \quad b \in \mathbb{R}.$$

Notation

We denote the set of complex numbers by \mathbb{C} .

So $z \in \mathbb{C}$ means that z is a complex number.

Arithmetic with complex numbers

Addition of complex numbers

We add complex numbers by adding their real and imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \quad a, b, c, d \in \mathbb{R}.$$

Multiplication of complex numbers

We multiply complex numbers by expanding and using $i^2 = -1$.

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Arithmetic with complex numbers

Examples

① $(2 - 3i) + (5 + i) =$

② $(2 - 3i)(5 + i) =$

③ $4(-4 + 5i) =$

④ **Q:** Which complex numbers z satisfy $z^2 = -4$?

A:

⑤ **Powers of i :**

▶ $i^0 =$

▶ $i^1 =$

▶ $i^2 =$

▶ $i^3 =$

▶ $i^4 =$

▶ $i^5 =$

Arithmetic with complex numbers

Subtraction of complex numbers

We define subtraction by

$$z - w = z + (-1)w, \quad z, w \in \mathbb{C}.$$

Example

$$(2 - 3i) - (4 - 7i) =$$

Complex conjugation

Definition (complex conjugation)

The **conjugate** of $z = a + bi$ ($a, b \in \mathbb{R}$) is

$$\bar{z} = a - bi$$

(we negate the imaginary part).

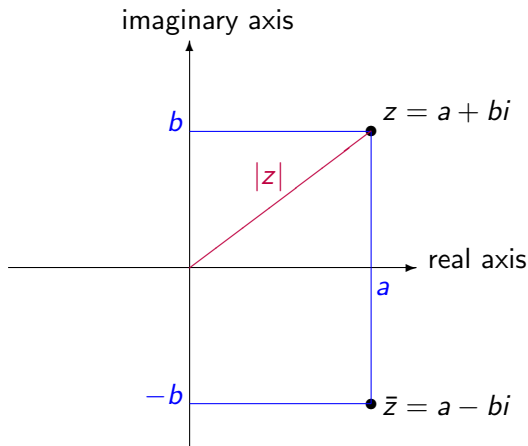
Examples

① $\overline{-5 + 2i} =$

② $\overline{3 - \frac{2}{5}i} =$

Geometric interpretation

Every complex number $a + bi$ corresponds to a point (a, b) (or vector) in the plane \mathbb{R}^2 .



$|z| = \sqrt{a^2 + b^2}$ is the distance of z from the origin (Pythagorean Theorem)

Geometric interpretation (cont.)

Complex number demonstration

<http://demonstrations.wolfram.com/ComplexNumber/>

Absolute values

If $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2 \geq 0.$$

Definition (absolute value or modulus)

If $z \in \mathbb{C}$, then the **absolute value** (or **modulus**) of z is

$$|z| \stackrel{\text{def}}{=} \sqrt{z\bar{z}}.$$

So if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}.$$

$|z|$ is always a nonnegative number (the distance from the origin to z , viewed as a point in the plane).

Note

If z is a real number, then $z = a + 0i$ and so

$$|z| = \sqrt{a^2}$$

which is the usual absolute value of a real number.

So our new definition agrees with the usual one for the special case of real numbers.

Examples

① If $z = -4 + 5i$, then

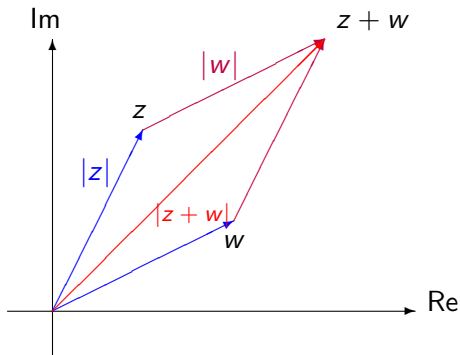
$$|z| =$$

② If $z = 3 + 4i$, then

$$|z| =$$

Geometric interpretation (cont.)

Addition of complex numbers corresponds to vector addition.



Recall: The length of one side of a triangle is less than the sum of lengths of the other two sides (the shortest distance between two points is a straight line). Thus

$$|z + w| \leq |z| + |w|.$$

This is called **the triangle inequality**.

Geometric interpretation (cont.)

Demonstration of addition of complex numbers

<http://demonstrations.wolfram.com/ComplexAddition/>

Some properties of complex numbers

If z and w are complex numbers, then

- 1 $\bar{z} = z$ iff $z \in \mathbb{R}$,
- 2 $\overline{w + z} = \bar{w} + \bar{z}$,
- 3 $\overline{wz} = \bar{w}\bar{z}$ (so, if $r \in \mathbb{R}$, then $\overline{rz} = r\bar{z} = r\bar{z}$),
- 4 $z\bar{z} = |z|^2 \geq 0$,
- 5 $|wz| = |w||z|$,
- 6 **triangle inequality:** $|w + z| \leq |w| + |z|$.

Multiplicative inverses and division

We've defined addition, subtraction and multiplication of complex numbers.

What about division?

Division is related to multiplicative inverses. For instance, if $a, b \in \mathbb{R}$, $b \neq 0$, then

$$a \div b = \frac{a}{b} = a \cdot \frac{1}{b} = a \cdot b^{-1}.$$

Dividing a complex number by a real number is easy: If $a, b, c \in \mathbb{R}$, $c \neq 0$, and $z = a + bi$, then

$$\frac{z}{c} =$$

Multiplicative inverses and division (cont.)

If $z \neq 0$, then $|z| > 0$ and

$$z \cdot \frac{\bar{z}}{|z|^2} = \frac{z\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

and so we write

$$\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2}.$$

This is the **multiplicative inverse** of z .

Example

If $z = 3 - 4i$, then

$$\frac{1}{z} = z^{-1} =$$

Multiplicative inverses and division (cont.)

Definition (quotients)

If $z, w \in \mathbb{C}$ and $w \neq 0$, then we define

$$\frac{z}{w} = z \cdot \frac{1}{w}.$$

Example

Suppose $w = 2 - 3i$ and $z = 5 + i$. Then

$$\frac{w}{z} =$$

Geometric interpretation of multiplication of complex numbers

Demonstration of multiplication of complex numbers

<http://demonstrations.wolfram.com/ComplexMultiplication/>

Example

Simplify

$$\frac{3 + 2i}{-4 - 3i}$$

That is, write it in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution:

$$\frac{3 + 2i}{-4 - 3i} =$$

Complex numbers in linear algebra

- For everything we've done so far, we can use complex numbers as the scalars instead of real numbers.

For example:

- ▶ linear equations/systems can involve complex numbers,
 - ▶ matrices can have complex numbers as entries,
 - ▶ we have discuss vectors in \mathbb{C}^n – these are just $n \times 1$ matrices with complex number entries,
 - ▶ we can talk about subspaces of \mathbb{C}^n instead of \mathbb{R}^n .
- We will see that certain statements/theorems are **only** true if we work with complex numbers.

Examples

1

$$\begin{bmatrix} 3 & i \\ 0 & -i \end{bmatrix} \begin{bmatrix} -1 & 2i \\ 0 & 5i \end{bmatrix} =$$

2

$$\text{Span} \left\{ \begin{bmatrix} 2i \\ 3+i \\ -4i \end{bmatrix}, \begin{bmatrix} 4 \\ -2-i \\ 4+7i \end{bmatrix} \right\} \text{ is}$$

3 Row reduction involving complex numbers:

$$\begin{bmatrix} 1 & i & -2 \\ -i & 1 & 3+2i \\ 2i & -2 & 6-4i \end{bmatrix}$$

Example

Solve the following equation for z :

$$(z + 3 - 2i)(1 + i) = 5$$

Solution:

Fundamental Theorem of Algebra

One of the most important properties of the complex numbers is that there are beautiful mathematical theorems that are **only** true if you work with complex numbers.

Fundamental Theorem of Algebra

Every non-constant single-variable polynomial with complex coefficients has at least one complex root.

Remember, that this was **not** true if we only considered real numbers.

Example

- Consider the polynomial $f(x) = x^2 + 1$.



Complex numbers and electronics (aside)

Basic electric circuits contain voltage sources (battery or power outlet), resistors, capacitors, and inductors.

- If your voltage source is **alternating** (like the voltage in your home), then
- the effect of a resistor depends on the current running through it, and
 - the effect of an inductor depends on the **rate of change** of the current running through it.

The equations to describe a circuit can get quite complicated. However, if we use **complex numbers**, then the roll of resistors and inductors is basically the same except:

- resistors have “impedances” that are real numbers, and
- inductors have “impedances” that are imaginary numbers!

Moral: Working with complex numbers **greatly** simplifies the problem.

Weekend problem

Next time

For next time: Read Sections EE, PEE.

- Eigenvectors.
- Eigenvalues.
- These are topics where complex numbers are essential.