MAT 1302B – Mathematical Methods II

Alistair Savage

Mathematics and Statistics
University of Ottawa

Winter 2015 – Lecture 15

Announcements

Second Midterm Exam:

- Handed back in DGDs this week.
- Grades posted on Blackboard Learn.

Today:

- Determinants and their properties.
- Relation between determinants and matrix inverses.

Review – Determinants of 1×1 and 2×2 matrices

Definition (determinant of a 1×1 matrix)

$$\det \begin{bmatrix} a \end{bmatrix} = a$$

Definition (determinant of a 2×2 matrix)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Review

Definition

Suppose A is an $n \times n$ matrix and $1 \leq i, j \leq n$. Then A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and j-th column of A.

Example

$$A = \begin{bmatrix} 2 & 5 & -8 & 7 & 4 \\ -1 & 0 & 3 & 2 & 1 \\ 8 & 1 & 5 & 2 & -10 \\ 0 & 0 & 1 & -3 & 5 \\ 4 & 0 & -8 & 7 & 6 \end{bmatrix}, \quad A_{4,5} = \begin{bmatrix} 2 & 5 & -8 & 7 \\ -1 & 0 & 3 & 2 \\ 8 & 1 & 5 & 2 \\ 4 & 0 & -8 & 7 \end{bmatrix}$$

Review – notation/terminology

1. We often use vertical bars to denote a determinant. That is, if $A = \left[a_{ij}\right]$, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

2. If $A = \left[a_{ij}\right]$, the (i,j)-cofactor of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Review – computing determinants

Theorem

If A is an $n \times n$ matrix, then

- $$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

  for any i such that $1 \leq i \leq n$ (cofactor expansion along i-th row), and

- $$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

  for any j such that $1 \leq j \leq n$ (cofactor expansion along j-th column).

Important point:

- We can expand along any row or column.
- So we choose rows or columns that make the computation easy (e.g. rows/columns that have a lot of zeros).

Review – sign in a cofactor expansion

Note: the sign of $(-1)^{i+j}$ in cofactor expansions gives signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Example

$$\begin{vmatrix} 1 & 2 & 18 & 0 \\ 4 & -1 & 7 & 1 \\ 0 & 0 & 3 & 0 \\ 3 & 1 & 10 & 2 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 & 0 \\ 4 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} \quad \text{(expanded along 3rd row)}$$

$$= 3\left((-1) \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} \right) \quad \begin{pmatrix}\text{expanded along} \\ \text{3rd column}\end{pmatrix}$$

$$= 3\Big(-\big(1 \cdot 1 - 2 \cdot 3\big) + 2\big(1(-1) - 2 \cdot 4\big)\Big)$$

$$= 3(-(-5) + 2(-9))$$

$$= 3(-13)$$

$$= -39$$

Determinant demonstration

Demonstration of computing a determinant by cofactor expansion

http://demonstrations.wolfram.com/33DeterminantsByExpansion/

Triangular matrices

Definition (triangular matrices)

Suppose A is a square matrix.

- If A has all zeros below the main diagonal, it is upper triangular.
- If A has all zeros above the main diagonal, it is lower triangular.
- A is triangular if it is upper or lower triangular.

Examples

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -3 & 5 & 6 \\ 0 & 0 & 8 & 3 & 2 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 7 & -8 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & -1 & 0 & 0 \\ 2 & -1 & 5 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

lower triangular

Determinants of triangular matrices

Theorem

If A is triangular, then $\det A$ is the product of the entries on the main diagonal.

Examples

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -3 & 5 & 6 \\ 0 & 0 & 8 & 3 & 2 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 0 & 7 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 7 & -8 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & -1 & 0 & 0 \\ 2 & -1 & 5 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 4$$

Row operations and determinants

Recall: Remember that row reduction involves 3 row operations.

Theorem

Suppose A is a square matrix.

1. If B is obtained from A by adding a multiple of one row of A to another row (row replacement), then $\det B = \det A$.
2. If B is obtained from A by interchanging two rows, then $\det B = -\det A$.
3. If B is obtained from A by multiplying a row of A by a scalar k, then $\det B = k \cdot \det A$.

Example

Compute $\det A$ if

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 4 \\ 4 & 3 & 0 \end{bmatrix}.$$

Solution: Recall that the determinant of a triangular matrix is the product of entries on the main diagonal.

$$\det A = \begin{vmatrix} 2 & 1 & -1 \\ -2 & -1 & 4 \\ 4 & 3 & 0 \end{vmatrix} \overset{R_1+R_2}{=} \begin{vmatrix} 2 & 1 & -1 \\ 0 & 0 & 3 \\ 4 & 3 & 0 \end{vmatrix}$$

$$\overset{-2R_1+R_3}{=} \begin{vmatrix} 2 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=} - \begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = -2 \cdot 1 \cdot 3 = -6$$

Common confusion

If we perform the operation multiply a row by the scalar c, then:

determinant after the operation $= c \cdot$ (determinant before the operation).

Thus,

determinant before the operation $= \frac{1}{c} \cdot$ (determinant after the operation).

Example

$$\underbrace{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 7 \\ -5 & 4 & 3 \end{vmatrix}}_{\text{before operation}} = \frac{1}{5} \underbrace{\begin{vmatrix} 5 & -5 & 5 \\ 2 & 3 & 7 \\ -5 & 4 & 3 \end{vmatrix}}_{\text{after operation}}, \qquad c = 5$$

Another example

$$\begin{vmatrix} 3 & -3 & 6 & 0 \\ 2 & 1 & 2 & 1 \\ -3 & 3 & -6 & 4 \\ -2 & 2 & -3 & -1 \end{vmatrix} \overset{\frac{1}{3}R_1}{=} 3 \begin{vmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 2 & 1 \\ -3 & 3 & -6 & 4 \\ -2 & 2 & -3 & -1 \end{vmatrix}$$

$$\overset{\substack{-2R_1+R_2 \\ 3R_1+R_3 \\ 2R_1+R_4}}{=} 3 \begin{vmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{vmatrix} \overset{R_3 \leftrightarrow R_4}{=} -3 \begin{vmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= -3 \cdot 1 \cdot 3 \cdot 1 \cdot 4 = -36$$

Note: We can always reduce a matrix to echelon form using row replacements and row interchanges (only needing scaling to reduce to RREF).

Note: A square echelon matrix is always triangular.

If U is an echelon matrix obtained from A by using only interchange and row replacement, then

$$\det A = (-1)^r \det U,$$

where

- r is the number of interchanges needed to reduce A to U, and
- $\det U = u_{11}u_{22}\dots u_{nn}$ (product of diagonal entries).

Note:

- If A is invertible, the diagonal entries $u_{11},\dots,u_{nn}$ are all pivots (hence nonzero).
- Otherwise, at least one diagonal entry is zero and so $\det U = 0$.

Determinants and inverses

Therefore,

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of nonzero pivots in } U) & \text{if } A \text{ is invertible,} \\ 0 & \text{if } A \text{ is not invertible.} \end{cases}$$

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

Notes:

1. Computation of a determinant using row operations as above is more efficient (in general) than expansion along rows/columns.
2. However, sometimes a combination of the two techniques is quickest.

Example

Compute $\det A$ where

$$A = \begin{bmatrix} 0 & 2 & -1 & 5 \\ 3 & 10 & -7 & 5 \\ -3 & -8 & 6 & 8 \\ 0 & 0 & -3 & 1 \end{bmatrix}.$$

Solution:

$$\det A \overset{R_2+R_3}{=} \begin{vmatrix} 0 & 2 & -1 & 5 \\ 3 & 10 & -7 & 5 \\ 0 & 2 & -1 & 13 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -3 \begin{vmatrix} 2 & -1 & 5 \\ 2 & -1 & 13 \\ 0 & -3 & 1 \end{vmatrix} \quad \begin{pmatrix}\text{expanded along} \\ \text{1st col}\end{pmatrix}$$

$$\overset{-R_1+R_2}{=} -3 \begin{vmatrix} 2 & -1 & 5 \\ 0 & 0 & 8 \\ 0 & -3 & 1 \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{=} 3 \begin{vmatrix} 2 & -1 & 5 \\ 0 & -3 & 1 \\ 0 & 0 & 8 \end{vmatrix}$$

$$= 3 \cdot 2 \cdot (-3) \cdot 8 = -144$$

Another example

Question: Suppose A is an $n \times n$ matrix and $c \in \mathbb{R}$. How are

$$\det A \qquad \text{and} \qquad \det cA$$

related?

Answer:

- cA is obtained from A by multiplying each of the n rows by c.
- Multiplying a single row by c multiplies the determinant by c.

Thus

$$\det cA = c^n \det A.$$

Examples

We know

$$\det cA = c^n \det A \quad \text{if } A \text{ is } n \times n.$$

Example 1

$$\det(2I_4) = 2^4 \det I_4 = 2^4 \cdot 1 = 2^4$$

Example 2

$$\det(-I_n) = (-1)^n \det I_n = \begin{cases} +1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Determinants and transpose

Theorem

If A is a square matrix, then

$$\det A = \det A^T.$$

Column operations

- Rows in A^T correspond to columns in A.
- Thus, previous theorem relating row operations to determinants is also true if we replace the word "row" everywhere by "column".

Determinants and column operations

Theorem

Suppose A is a square matrix.

1. If B is obtained from A by adding a multiple of one column of A to another column (column replacement), then $\det B = \det A$.
2. If B is obtained from A by interchanging two columns, then $\det B = -\det A$.
3. If B is obtained from A by multiplying a column of A by a scalar k, then $\det B = k \cdot \det A$.

Example

$$\begin{vmatrix} 2 & 0 & 4 & 10 \\ -4 & 3 & -8 & 5 \\ 5 & 8 & 10 & -7 \\ 7 & -2 & 14 & 2 \end{vmatrix} \overset{-2C_1+C_3}{=} \begin{vmatrix} 2 & 0 & 0 & 10 \\ -4 & 3 & 0 & 5 \\ 5 & 8 & 0 & -7 \\ 7 & -2 & 0 & 2 \end{vmatrix}$$

$$= 0 \qquad \text{(expanded along 3rd column)}$$

Important note: Only use column operations when computing determinants, **NOT** when row reducing.

Example

Suppose

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3.$$

Then

$$\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g-d & h-e & i-f \end{vmatrix} = 2 \cdot 3 = 6,$$

since we obtained the above matrix from the original one by multiplying row 1 by 2 and adding -1 times row 2 to row 3.

Similarly,

$$\begin{vmatrix} -a & c & b \\ -d & f & e \\ -g & i & h \end{vmatrix} = (-1)(-1)(3) = 3,$$

since we obtained the above matrix from the original one by multiplying column 1 by -1 and swapping columns 2 and 3.

Determinants and matrix products

Theorem (multiplicative property of determinants)

If A and B are $n \times n$ matrices, then

$$\det AB = (\det A)(\det B).$$

Example

If $A = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$, then

$$\det A = 4 \cdot 2 - 3 \cdot 5 = -7, \qquad \det B = (-1) \cdot 4 - 2 \cdot (-3) = 2$$

Also,

$$AB = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -13 & 20 \\ -11 & 18 \end{bmatrix},$$

and so

$$\det AB = (-13) \cdot 18 - 20 \cdot (-11) = -14 = (-7) \cdot 2 = (\det A)(\det B).$$

Determinants and matrix inverses

Suppose A is an invertible matrix.

Then $\det A \neq 0$.

We have

$$(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1.$$

Proposition

If A is an invertible matrix (so $\det A \neq 0$), then

$$\det A^{-1} = \frac{1}{\det A}.$$

Example

Suppose A, B, C are 3×3 matrices with $\det A = 2$, $\det B = -3$ and

$$A^2 C^T B = BA.$$

Find $\det C$. Is C invertible? If so, find $\det C^{-1}$.

Solution: We have

$$\begin{aligned} \det(A^2 C^T B) &= \det(BA) \\ \implies (\det A)^2 (\det C^T)(\det B) &= (\det B)(\det A) \\ \implies 2^2 (\det C^T)(-3) &= (-3)2 \\ \implies -12(\det C^T) &= -6 \\ \implies \det C^T = \frac{1}{2} &\implies \det C = \det C^T = \frac{1}{2}. \end{aligned}$$

Since $\det C \neq 0$, C is invertible and

$$\det C^{-1} = \frac{1}{\det C} = 2.$$

Determinants and matrix inverses

We can use the property

$$\det A \neq 0 \iff A \text{ is invertible}$$

to prove some properties of inverses.

Examples:

1. We saw before that if A and B are invertible $n \times n$ matrices, then AB is invertible (and its inverse is $B^{-1}A^{-1}$). We can see this in another way now:
   - A, B invertible $\implies \det A, \det B \neq 0$
   - Thus $\det AB = (\det A)(\det B) \neq 0$.
   - So AB is invertible.
2. Suppose one of A and B is not invertible.
   - Then $\det A = 0$ or $\det B = 0$.
   - Thus $\det AB = (\det A)(\det B) = 0$.
   - So AB is not invertible.

Geometric interpretation of determinants

Determinants and areas/volumes

- 2 dimensions: The absolute value of the determinant of a 2×2 matrix $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$ is the area of the parallelogram determined by the vectors $\vec{a}_1$ and $\vec{a}_2$.
- 3 dimensions: The absolute value of the determinant of a 3×3 matrix $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ is the volume of the parallelepiped determined by the vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$.

Demonstration of geometric interpretation of determinants

http://demonstrations.wolfram.com/DeterminantsSeenGeometrically/

Weekend problem – last time

Bill and Pete are two hockey players.

2007 Season:

- Bill averages 1 point per game.
- Pete averages 0.5 points per game

2008 Season:

- Bill averages 3 points per game.
- Pete averages 2 points per game.

Question:

- Consider the 2007 and 2008 seasons together.
- Is it necessarily true that Bill's average points per game over the two seasons is higher than Pete's?

Weekend problem – solution

Answer: NO!

For instance, the following could happen.

2007 Season:

- Bill gets 50 points in 50 games (averages 1 point per game).
- Pete gets 10 points in 20 games (averages 0.5 points per game).

2008 Season:

- Bill gets 30 points in 10 games (averages 3 points per game).
- Pete gets 100 points in 50 games (averages 2 points per game).

2007 and 2008 Seasons combined:

- Bill has 80 points in 60 games (averaged 1.33 points per game).
- Pete has 110 points in 70 games (averaged 1.57 points per game).

Next time

For next time: Read Section CNO.

- Complex numbers
- Imaginary numbers