

# MAT 1302B – Mathematical Methods II

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

# Announcements

## Second Midterm Exam:

- Handed back in DGDs this week.
- Grades posted on Blackboard Learn.

## Today:

- Determinants and their properties.
- Relation between determinants and matrix inverses.

## Review – Determinants of $1 \times 1$ and $2 \times 2$ matrices

Definition (determinant of a  $1 \times 1$  matrix)

$$\det [a] =$$

Definition (determinant of a  $2 \times 2$  matrix)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

# Review

## Definition

Suppose  $A$  is an  $n \times n$  matrix and  $1 \leq i, j \leq n$ . Then  $A_{ij}$  is the  $(n-1) \times (n-1)$  matrix obtained by removing the  $i$ -th row and  $j$ -th column of  $A$ .

## Example

$$A = \begin{bmatrix} 2 & 5 & -8 & 7 & 4 \\ -1 & 0 & 3 & 2 & 1 \\ 8 & 1 & 5 & 2 & -10 \\ 0 & 0 & 1 & -3 & 5 \\ 4 & 0 & -8 & 7 & 6 \end{bmatrix}, \quad A_{4,5} =$$

## Review – notation/terminology

- ① We often use vertical bars to denote a determinant. That is, if  $A = [a_{ij}]$ , then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

- ② If  $A = [a_{ij}]$ , the  $(i, j)$ -cofactor of  $A$  is

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

# Review – computing determinants

## Theorem

If  $A$  is an  $n \times n$  matrix, then

- $$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

for any  $i$  such that  $1 \leq i \leq n$  (cofactor expansion along  $i$ -th row), and

- $$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any  $j$  such that  $1 \leq j \leq n$  (cofactor expansion along  $j$ -th column).

## Important point:

- We can expand along **any** row or column.
- So we choose rows or columns that make the computation easy (e.g. rows/columns that have a lot of zeros).

## Review – sign in a cofactor expansion

**Note:** the sign of  $(-1)^{i+j}$  in cofactor expansions gives signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

## Example

$$\begin{vmatrix} 1 & 2 & 18 & 0 \\ 4 & -1 & 7 & 1 \\ 0 & 0 & 3 & 0 \\ 3 & 1 & 10 & 2 \end{vmatrix} =$$



# Determinant demonstration

Demonstration of computing a determinant by cofactor expansion

<http://demonstrations.wolfram.com/33DeterminantsByExpansion/>

# Triangular matrices

## Definition (triangular matrices)

Suppose  $A$  is a square matrix.

- If  $A$  has all zeros below the main diagonal, it is **upper triangular**.
- If  $A$  has all zeros above the main diagonal, it is **lower triangular**.
- $A$  is **triangular** if it is upper or lower triangular.

## Examples

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -3 & 5 & 6 \\ 0 & 0 & 8 & 3 & 2 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 7 & -8 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & -1 & 0 & 0 \\ 2 & -1 & 5 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Determinants of triangular matrices

## Theorem

If  $A$  is triangular, then  $\det A$  is the product of the entries on the main diagonal.

## Examples

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -3 & 5 & 6 \\ 0 & 0 & 8 & 3 & 2 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 0 & 7 \end{vmatrix} = , \quad \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 7 & -8 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & -1 & 0 & 0 \\ 2 & -1 & 5 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} =$$

# Row operations and determinants

**Recall:** Remember that row reduction involves 3 row operations.

## Theorem

Suppose  $A$  is a square matrix.

- 1 If  $B$  is obtained from  $A$  by adding a multiple of one row of  $A$  to another row (row replacement), then  $\det B = \det A$ .
- 2 If  $B$  is obtained from  $A$  by interchanging two rows, then  $\det B = -\det A$ .
- 3 If  $B$  is obtained from  $A$  by multiplying a row of  $A$  by a scalar  $k$ , then  $\det B = k \cdot \det A$ .

## Example

Compute  $\det A$  if

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 4 \\ 4 & 3 & 0 \end{bmatrix}.$$

**Solution:** Recall that the determinant of a triangular matrix is the product of entries on the main diagonal.

$$\det A = \begin{vmatrix} 2 & 1 & -1 \\ -2 & -1 & 4 \\ 4 & 3 & 0 \end{vmatrix}$$

## Common confusion

If we perform the operation **multiply a row by the scalar  $c$** , then:

determinant **after** the operation =  $c \cdot$  (determinant **before** the operation).

Thus,

determinant **before** the operation =  $\frac{1}{c} \cdot$  (determinant **after** the operation).

### Example

## Another example

$$\left| \begin{array}{cccc} 3 & -3 & 6 & 0 \\ 2 & 1 & 2 & 1 \\ -3 & 3 & -6 & 4 \\ -2 & 2 & -3 & -1 \end{array} \right|$$

**Note:** We can always reduce a matrix to echelon form using row replacements and row interchanges (only needing scaling to reduce to RREF).

**Note:** A square echelon matrix is always triangular.

If  $U$  is an echelon matrix obtained from  $A$  by using only interchange and row replacement, then

$$\det A = (-1)^r \det U,$$

where

- $r$  is the number of interchanges needed to reduce  $A$  to  $U$ , and
- $\det U = u_{11}u_{22} \dots u_{nn}$  (product of diagonal entries).

**Note:**

- If  $A$  is invertible, the diagonal entries  $u_{11}, \dots, u_{nn}$  are all pivots (hence nonzero).
- Otherwise, at least one diagonal entry is zero and so  $\det U = 0$ .



# Determinants and inverses

Therefore,

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U) & \text{if } A \text{ is invertible,} \\ 0 & \text{if } A \text{ is not invertible.} \end{cases}$$

## Theorem

A square matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

## Notes:

- 1 Computation of a determinant using row operations as above is more efficient (in general) than expansion along rows/columns.
- 2 However, sometimes a combination of the two techniques is quickest.

## Example

Compute  $\det A$  where

$$A = \begin{bmatrix} 0 & 2 & -1 & 5 \\ 3 & 10 & -7 & 5 \\ -3 & -8 & 6 & 8 \\ 0 & 0 & -3 & 1 \end{bmatrix}.$$

Solution:

## Another example

**Question:** Suppose  $A$  is an  $n \times n$  matrix and  $c \in \mathbb{R}$ . How are

$\det A$       and       $\det cA$

related?

**Answer:**

# Examples

We know

## Example 1

$$\det(2I_4) =$$

## Example 2

$$\det(-I_n) =$$

# Determinants and transpose

## Theorem

If  $A$  is a square matrix, then

$$\det A = \det A^T.$$

## Column operations

- Rows in  $A^T$  correspond to columns in  $A$ .
- Thus, previous theorem relating row operations to determinants is also true if we replace the word “row” everywhere by “column”.

# Determinants and column operations

## Theorem

Suppose  $A$  is a square matrix.

- 1 If  $B$  is obtained from  $A$  by adding a multiple of one column of  $A$  to another column (column replacement), then  $\det B = \det A$ .
- 2 If  $B$  is obtained from  $A$  by interchanging two columns, then  $\det B = -\det A$ .
- 3 If  $B$  is obtained from  $A$  by multiplying a column of  $A$  by a scalar  $k$ , then  $\det B = k \cdot \det A$ .

## Example

$$\begin{vmatrix} 2 & 0 & 4 & 10 \\ -4 & 3 & -8 & 5 \\ 5 & 8 & 10 & -7 \\ 7 & -2 & 14 & 2 \end{vmatrix}$$

**Important note:** Only use column operations when computing determinants, **NOT** when row reducing.

## Example

Suppose

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3.$$

Then

$$\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g-d & h-e & i-f \end{vmatrix} =$$

Similarly,

$$\begin{vmatrix} -a & c & b \\ -d & f & e \\ -g & i & h \end{vmatrix} =$$



# Determinants and matrix products

## Theorem (multiplicative property of determinants)

If  $A$  and  $B$  are  $n \times n$  matrices, then

$$\det AB = (\det A)(\det B).$$

## Example

If  $A = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$ , then

$$\det A =$$

$$\det B =$$

Also,

$$AB = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} =$$

and so

$$\det AB =$$

## Determinants and matrix inverses

Suppose  $A$  is an invertible matrix.

Then  $\det A \neq 0$ .

We have

$$(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1.$$

### Proposition

If  $A$  is an invertible matrix (so  $\det A \neq 0$ ), then

$$\det A^{-1} = \frac{1}{\det A}.$$

## Example

Suppose  $A, B, C$  are  $3 \times 3$  matrices with  $\det A = 2$ ,  $\det B = -3$  and

$$A^2 C^T B = BA.$$

Find  $\det C$ . Is  $C$  invertible? If so, find  $\det C^{-1}$ .

**Solution:**

# Determinants and matrix inverses

We can use the property

$$\det A \neq 0 \iff A \text{ is invertible}$$

to prove some properties of inverses.

## Examples:

- 1 We saw before that if  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $AB$  is invertible (and its inverse is  $B^{-1}A^{-1}$ ). We can see this in another way now:



- 2 Suppose one of  $A$  and  $B$  is **not** invertible.



# Geometric interpretation of determinants

## Determinants and areas/volumes

- **2 dimensions:** The absolute value of the determinant of a  $2 \times 2$  matrix  $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$  is the area of the parallelogram determined by the vectors  $\vec{a}_1$  and  $\vec{a}_2$ .
- **3 dimensions:** The absolute value of the determinant of a  $3 \times 3$  matrix  $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$  is the volume of the parallelepiped determined by the vectors  $\vec{a}_1$ ,  $\vec{a}_2$  and  $\vec{a}_3$ .

## Demonstration of geometric interpretation of determinants

<http://demonstrations.wolfram.com/DeterminantsSeenGeometrically/>

## Weekend problem – last time

Bill and Pete are two hockey players.

### 2007 Season:

- Bill averages 1 point per game.
- Pete averages 0.5 points per game

### 2008 Season:

- Bill averages 3 points per game.
- Pete averages 2 points per game.

### Question:

- Consider the 2007 and 2008 seasons together.
- Is it necessarily true that Bill's average points per game over the two seasons is higher than Pete's?

# Weekend problem – solution

## Next time

For next time: Read Section CNO.

- Complex numbers
- Imaginary numbers