MAT 1302B – Mathematical Methods II

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Announcements

Second Midterm:

- Handed back in DGDs next week.

Today:

- Brief review of subspaces and bases.
- Determinants.

Review – subspaces of $\mathbb{R}^n$

Definition (Subspace)

A subspace of $\mathbb{R}^n$ is any subset H of $\mathbb{R}^n$ that satisfies the following 3 conditions:

1. $\vec{0} \in H$.
2. If $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$.
3. If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$.

Definition (Basis)

Suppose H is a subspace of $\mathbb{R}^n$. A basis of H is a linearly independent set of vectors in H that span H.

In other words, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis of H if

- $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, and
- $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = H$.

Review – important examples of subspaces

1. The set $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.
2. $\mathbb{R}^n$ is a subspace of itself.
3. Given vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, their span

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$$

is a subspace of $\mathbb{R}^n$.

4. Column spaces: If A is an $m \times n$ matrix, then $\text{Col}\, A$ is the span of the columns of A and hence is a subspace of $\mathbb{R}^m$.

Col A is also the set of $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ has a solution.

5. Null spaces: If A is an $m \times n$ matrix, $\text{Nul}\, A$ is the set of solutions of the equation $A\vec{x} = \vec{0}$. That is,

$$\text{Nul}\, A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

Nul A is a subspace of $\mathbb{R}^n$.

Review – basis of a null space

How to find a basis of the null space of a matrix

To find a basis of the null space of a matrix A, we:

1. Solve the homogeneous equation $A\vec{x} = \vec{0}$ by row reduction.
2. Write the general solution in vector notation, as a span of vectors. E.g.
$$\vec{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 6 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad x_2, x_3, x_5 \in \mathbb{R}.$$
3. The set of vectors appearing in this form of the solution is a basis for $\operatorname{Nul} A$. E.g. (in above example)
$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \quad \text{is a basis of } \operatorname{Nul} A.$$

Review – basis of a column space

Theorem

The pivot columns of a matrix form a basis for its column space.

Procedure for finding a basis of a column space

To find a basis for the column space of a matrix A we:

1. Row reduce the matrix to EF to determine which columns are the pivot columns. (You don't need to go all the way to RREF!)
2. A basis for the column space is the set containing the pivot columns of the original matrix A.

Note: You must use the pivot columns of the original matrix and not the matrix you get after row reducing.

Review – basis of a span

Problem:

- You're given a collection of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$.
- You're asked to find a basis for their span:

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$$

Solution:

- Form a matrix with the vectors as columns:

$$A = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$$

- Then $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \operatorname{Col} A$.
- So we need to find a basis for $\operatorname{Col} A$.
- Row reduce A to find the pivot columns.
- The set of pivot columns of A is a basis of $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$.

Review – dimension

Theorem

Any two bases of a given subspace have the same number of vectors.

Definition (Dimension)

The dimension of a nonzero subspace H, denoted $\dim H$, is the number of vectors in any basis of H.

We define $\dim\{\vec{0}\} = 0$. The emptyset $\varnothing$ is a basis of $\{\vec{0}\}$.

Examples

- Lines through the origin are one-dimensional subspaces.
- Planes containing the origin are two-dimensional subspaces.
- $\mathbb{R}^n$ is n-dimensional.

Review – rank and nullity

Rank of a matrix

- rank A is the dimension of the column space of A.
- rank A is the number of pivot columns/positions of A.

Nullity of a matrix

- Nullity of A is the dimension of Nul A.
- Nullity of A is the number of non-pivot columns in A.
- If A has n columns, then

$$\dim \operatorname{Nul} A = n - \text{ pivot positions in } A.$$

Rank Theorem

If a matrix A has n columns, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Recall – matrix inverses

Definition (Inverse of a matrix)

An $n \times n$ (square) matrix A is invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n.$$

We call C an inverse of A (it is unique and denoted A^{-1}).

Theorem (Inverses of 2×2 matrices)

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $ad - bc = 0$, then A is not invertible.

Recall – matrix inverses

Definition (determinant of a 2×2 matrix)

The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det A = ad - bc.$$

So A is invertible if and only if $\det A \neq 0$.

Today's goal

We want to define a determinant for arbitrary square matrices with the same property. That is, we want

$$A \text{ invertible} \iff \det A \neq 0.$$

1×1 matrices

For 1×1 matrices,

$$\begin{bmatrix} a_{11} \end{bmatrix} \text{ invertible} \iff a_{11} \neq 0$$

since if $a_{11} \neq 0$, then

$$\begin{bmatrix} a_{11} \end{bmatrix} \begin{bmatrix} \frac{1}{a_{11}} \end{bmatrix} = \begin{bmatrix} \frac{1}{a_{11}} \end{bmatrix} \begin{bmatrix} a_{11} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} = I_1.$$

Definition (determinant of a 1×1 matrix)

We define

$$\det \begin{bmatrix} a_{11} \end{bmatrix} = a_{11}.$$

Definition

Suppose A is an $n \times n$ matrix and $1 \leq i, j \leq n$. Then A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and j-th column of A.

Example

$$A = \begin{bmatrix} 2 & 5 & -8 & 7 & 4 \\ -1 & 0 & 3 & 2 & 1 \\ 8 & 1 & 5 & 2 & -10 \\ 0 & 0 & 1 & -3 & 5 \\ 4 & 0 & -8 & 7 & 6 \end{bmatrix}, \quad A_{3,4} = \begin{bmatrix} 2 & 5 & -8 & 4 \\ -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 5 \\ 4 & 0 & -8 & 6 \end{bmatrix}$$

Determinants

Definition (determinant)

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Notes

- This defines the determinant of a matrix in terms of determinants of smaller matrices.
- We continue until we reduce the computation to determinants of 2×2 or 1×1 matrices.

Example

Suppose

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{bmatrix}.$$

Then

$$\begin{aligned} \det A &= 2 \det \begin{bmatrix} -2 & 1 \\ 5 & -3 \end{bmatrix} - (-1) \det \begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix} + (-3) \det \begin{bmatrix} 4 & -2 \\ 0 & 5 \end{bmatrix} \\ &= 2\big((-2)(-3) - 5 \cdot 1\big) + \big(4(-3) - 0 \cdot 1\big) - 3\big(4 \cdot 5 - 0(-2)\big) \\ &= 2 \cdot 1 + (-12) - 3 \cdot 20 \\ &= -70. \end{aligned}$$

Relation to previous definition

Let's apply our formula to a 2×2 matrix:

$$\begin{aligned} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= a_{11} \det \begin{bmatrix} a_{22} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} \end{bmatrix} \\ &= a_{11} a_{22} - a_{12} a_{21} \end{aligned}$$

which is the same definition we gave before!

Notation/terminology

1. We often use vertical bars to denote a determinant. That is, if $A = \left[a_{ij}\right]$, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

2. If $A = \left[a_{ij}\right]$, the (i, j)-cofactor of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Thus

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

This is the cofactor expansion along the first row.

Computing determinants

Theorem

If A is an $n \times n$ matrix, then

- $$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

  for any i such that $1 \leq i \leq n$ (cofactor expansion along i-th row), and

- $$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

  for any j such that $1 \leq j \leq n$ (cofactor expansion along j-th column).

Important point:

- We can expand along any row or column.
- So we choose rows or columns that make the computation easy (e.g. rows/columns that have a lot of zeros).

Example

Consider the matrix from the previous example:

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{bmatrix}$$

Let's compute the determinant by cofactor expansion along the third row.

Then

$$\begin{aligned} \det A &= 0 \cdot \begin{vmatrix} -1 & -3 \\ -2 & 1 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} + (-3) \cdot \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} \\ &= -5\big(2 \cdot 1 - (-3)4\big) - 3\big(2(-2) - (-1)4\big) \\ &= -5(14) - 3(0) = -70 \end{aligned}$$

as before!

Note: Expanding along the third row was easier because it had a zero.

Another example

Compute $\det A$ if

$$A = \begin{bmatrix} 5 & 0 & 2 & 11 \\ 10 & 3 & 7 & -8 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 1 & -9 \end{bmatrix}.$$

Solution: We first expand along the second column:

$$\det A = 0 \cdot C_{12} + 3 \begin{vmatrix} 5 & 2 & 11 \\ 0 & 0 & 2 \\ 2 & 1 & -9 \end{vmatrix} + 0 \cdot C_{32} + 0 \cdot C_{42}.$$

Then we expand along the second row:

$$\det A = 3 \left(0 + 0 - 2 \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} \right)$$
$$= -6(5 \cdot 1 - 2 \cdot 2) = -6.$$

Note: the sign of $(-1)^{i+j}$ in cofactor expansions gives signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Another example

$$\begin{aligned}
\begin{vmatrix} 4 & 0 & 5 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ -1 & 2 & 1 & 1 \end{vmatrix} &= -1 \begin{vmatrix} 0 & 5 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 4 & 0 & 0 \\ 3 & 1 & 2 \\ -1 & 2 & 1 \end{vmatrix} \\
&= (-1)(-5) \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\
&= 5(1 \cdot 1 - 2 \cdot 2) + 4(1 \cdot 1 - 2 \cdot 2) \\
&= 5(-3) + 4(-3) \\
&= -27
\end{aligned}$$

Yet another example

$$d = \begin{vmatrix} 2 & 0 & -1 & -3 \\ 1 & 4 & 0 & 5 \\ 0 & 2 & -1 & 0 \\ -2 & 3 & 1 & 5 \end{vmatrix} = -2 \begin{vmatrix} 2 & -1 & -3 \\ 1 & 0 & 5 \\ -2 & 1 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \\ -2 & 3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 & -3 \\ 1 & 0 & 5 \\ -2 & 1 & 5 \end{vmatrix} = -1 \begin{vmatrix} -1 & -3 \\ 1 & 5 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix}$$
$$= -\big((-1)5 - (-3)1\big) - 5\big(2 \cdot 1 - (-1)(-2)\big) = 2$$

$$\begin{vmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \\ -2 & 3 & 5 \end{vmatrix} = 4 \begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}$$
$$= 4(2 \cdot 5 - (-3)(-2)) - 3(2 \cdot 5 - (-3)1) = -23$$

So $d = (-2)2 + (-1)(-23) = 19$

Triangular matrices

Definition (triangular matrices)

Suppose A is a square matrix.

- If A has all zeros below the main diagonal, it is upper triangular.
- If A has all zeros above the main diagonal, it is lower triangular.
- A is triangular if it is upper or lower triangular.

Examples

$$\begin{bmatrix} 2 & 7 & 8 & 9 & 10 \\ 0 & -1 & 4 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 & 0 \\ 7 & 7 & 10 & -3 & 0 & 0 \\ 8 & -3 & 10 & 6 & 5 & 0 \\ 3 & -3 & 5 & 7 & 8 & 3 \end{bmatrix}$$

lower triangular

Determinants of triangular matrices

Theorem

If A is triangular, then $\det A$ is the product of the entries on the main diagonal.

Justification of the theorem

Let's do the case where A is upper triangular (the lower triangular case is similar).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Justification of theorem (cont.)

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$ expand along 1st column

$$= a_{11} \begin{vmatrix} a_{22} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$ expand along 1st column

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & \cdots & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$ expand along 1st column

$$\vdots$$

$$= a_{11}a_{22}\cdots a_{nn}$$

Examples

1. $$\det \begin{bmatrix} 4 & 0 & 0 & 0 \\ 10 & \frac{1}{2} & 0 & 0 \\ 20 & -7 & 1 & 0 \\ -8 & 0 & 7 & 5 \end{bmatrix} = 4 \cdot \frac{1}{2} \cdot 1 \cdot 5 = 10$$

2. $$\begin{vmatrix} 8 & 7 & 9 & -3 & 4 & 5 & 0 \\ 0 & 2 & 5 & 6 & 3 & 4 & -2 \\ 0 & 0 & 0 & 2 & 3 & -4 & -5 \\ 0 & 0 & 0 & 7 & 8 & -9 & 11 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{vmatrix} = 0$$

Weekend problem

Bill and Pete are two hockey players.

2007 Season:

- Bill averages 1 point per game.
- Pete averages 0.5 points per game.

2008 Season:

- Bill averages 3 points per game.
- Pete averages 2 points per game.

Question:

- Consider the 2007 and 2008 seasons together.
- Is it necessarily true that Bill's average points per game over the two seasons is higher than Pete's?

Next time

For next time: Read Section PDM.

- Properties of determinants.
- Relation between determinants and invertibility of matrices.