

MAT 1302B – Mathematical Methods II

Alistair Savage

Mathematics and Statistics
University of Ottawa

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Announcements

Second Midterm:

- Handed back in DGDs next week.

Today:

- Brief review of subspaces and bases.
- Determinants.

Review – subspaces of \mathbb{R}^n

Definition (Subspace)

A **subspace** of \mathbb{R}^n is any subset H of \mathbb{R}^n that satisfies the following 3 conditions:

- 1 $\vec{0} \in H$.
- 2 If $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$.
- 3 If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$.

Definition (Basis)

Suppose H is a subspace of \mathbb{R}^n . A **basis** of H is a linearly independent set of vectors in H that span H .

In other words, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis of H if

- $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, and
- $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = H$.

Review – important examples of subspaces

- 1 The set $\{\vec{0}\}$ is a subspace of \mathbb{R}^n .
- 2 \mathbb{R}^n is a subspace of itself.
- 3 Given vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, their span

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$$

is a subspace of \mathbb{R}^n .

- 4 **Column spaces:** If A is an $m \times n$ matrix, then $\text{Col } A$ is the span of the columns of A and hence is a subspace of \mathbb{R}^m .

$\text{Col } A$ is also the set of $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ has a solution.

- 5 **Null spaces:** If A is an $m \times n$ matrix, $\text{Nul } A$ is the set of solutions of the equation $A\vec{x} = \vec{0}$. That is,

$$\text{Nul } A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

$\text{Nul } A$ is a subspace of \mathbb{R}^n .

Review – basis of a null space

How to find a basis of the null space of a matrix

To find a basis of the null space of a matrix A , we:

- 1 Solve the homogeneous equation $A\vec{x} = \vec{0}$ by row reduction.
- 2 Write the general solution in vector notation, as a span of vectors.
E.g.

$$\vec{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 6 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad x_2, x_3, x_5 \in \mathbb{R}.$$

- 3 The set of vectors appearing in this form of the solution is a basis for $\text{Nul } A$. E.g. (in above example)

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \quad \text{is a basis of } \text{Nul } A.$$

Review – basis of a column space

Theorem

The pivot columns of a matrix form a basis for its column space.

Procedure for finding a basis of a column space

To find a basis for the column space of a matrix A we:

- 1 Row reduce the matrix to EF to determine which columns are the pivot columns. (You **don't** need to go all the way to RREF!)
- 2 A basis for the column space is the set containing the pivot columns of the **original matrix** A .

Note: You must use the pivot columns of the **original matrix** and **not** the matrix you get after row reducing.

Review – basis of a span

Problem:

- You're given a collection of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$.
- You're asked to find a basis for their span:

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$$

Solution:

-
-
-
-
-

Review – dimension

Theorem

Any two bases of a given subspace have the same number of vectors.

Definition (Dimension)

The **dimension** of a nonzero subspace H , denoted $\dim H$, is the number of vectors in any basis of H .

We define $\dim\{\vec{0}\} = 0$.

Examples

- Lines through the origin are _____ subspaces.
- Planes containing the origin _____ subspaces.
- \mathbb{R}^n is _____.

Review – rank and nullity

Rank of a matrix

- rank A is _____
- rank A is _____

Nullity of a matrix

- Nullity of A is _____
- Nullity of A is _____
- If A has n columns, then

Rank Theorem

If a matrix A has n columns, then

Recall – matrix inverses

Definition (Inverse of a matrix)

An $n \times n$ (square) matrix A is **invertible** if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n.$$

We call C an **inverse** of A (it is unique and denoted A^{-1}).

Theorem (Inverses of 2×2 matrices)

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $ad - bc = 0$, then A is not invertible.

Recall – matrix inverses

Definition (determinant of a 2×2 matrix)

The **determinant** of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det A = ad - bc.$$

So A is invertible if and only if $\det A \neq 0$.

Today's goal

We want to define a determinant for arbitrary square matrices with the same property. That is, we want

$$A \text{ invertible} \iff \det A \neq 0.$$

1×1 matrices

For 1×1 matrices,

$$[a_{11}] \text{ invertible} \iff$$

since

Definition (determinant of a 1×1 matrix)

We define

$$\det [a_{11}] =$$

Definition

Suppose A is an $n \times n$ matrix and $1 \leq i, j \leq n$. Then A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing the i -th row and j -th column of A .

Example

$$A = \begin{bmatrix} 2 & 5 & -8 & 7 & 4 \\ -1 & 0 & 3 & 2 & 1 \\ 8 & 1 & 5 & 2 & -10 \\ 0 & 0 & 1 & -3 & 5 \\ 4 & 0 & -8 & 7 & 6 \end{bmatrix}, \quad A_{3,4} =$$

Determinants

Definition (determinant)

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

Notes

- This defines the determinant of a matrix in terms of determinants of smaller matrices.
- We continue until we reduce the computation to determinants of 2×2 or 1×1 matrices.

Example

Suppose

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{bmatrix}.$$

Then

$$\det A =$$

Relation to previous definition

Let's apply our formula to a 2×2 matrix:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Notation/terminology

- ① We often use vertical bars to denote a determinant. That is, if $A = [a_{ij}]$, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

- ② If $A = [a_{ij}]$, the (i, j) -cofactor of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Thus

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

This is the **cofactor expansion along the first row**.

Computing determinants

Theorem

If A is an $n \times n$ matrix, then

- $$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

for any i such that $1 \leq i \leq n$ (cofactor expansion along i -th row), and

- $$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

for any j such that $1 \leq j \leq n$ (cofactor expansion along j -th column).

Important point:

- We can expand along **any** row or column.
- So we choose rows or columns that make the computation easy (e.g. rows/columns that have a lot of zeros).

Example

Consider the matrix from the previous example:

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & -2 & 1 \\ 0 & 5 & -3 \end{bmatrix}$$

Let's compute the determinant by cofactor expansion along the third row.

Then

$$\det A =$$

Note: Expanding along the third row was easier because it had a zero.

Another example

Compute $\det A$ if

$$A = \begin{bmatrix} 5 & 0 & 2 & 11 \\ 10 & 3 & 7 & -8 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 1 & -9 \end{bmatrix}.$$

Solution: We first expand along the _____:

$$\det A =$$

Then we expand along the _____:

$$\det A =$$

$$=$$

Note: the sign of $(-1)^{i+j}$ in cofactor expansions gives signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Another example

$$\begin{vmatrix} 4 & 0 & 5 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ -1 & 2 & 1 & 1 \end{vmatrix} =$$

Yet another example

$$d = \begin{vmatrix} 2 & 0 & -1 & -3 \\ 1 & 4 & 0 & 5 \\ 0 & 2 & -1 & 0 \\ -2 & 3 & 1 & 5 \end{vmatrix} =$$

Triangular matrices

Definition (triangular matrices)

Suppose A is a square matrix.

- If A has all zeros below the main diagonal, it is **upper triangular**.
- If A has all zeros above the main diagonal, it is **lower triangular**.
- A is **triangular** if it is upper or lower triangular.

Examples

$$\begin{bmatrix} 2 & 7 & 8 & 9 & 10 \\ 0 & -1 & 4 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & -4 & 1 & 0 & 0 & 0 \\ 7 & 7 & 10 & -3 & 0 & 0 \\ 8 & -3 & 10 & 6 & 5 & 0 \\ 3 & -3 & 5 & 7 & 8 & 3 \end{bmatrix}$$

Determinants of triangular matrices

Theorem

If A is triangular, then $\det A$ is the product of the entries on the main diagonal.

Justification of the theorem

Let's do the case where A is upper triangular (the lower triangular case is similar).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Justification of theorem (cont.)

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

expand along 1st column

$$= a_{11} \begin{vmatrix} a_{22} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

expand along 1st column

$$= a_{11} a_{22} \begin{vmatrix} a_{33} & \cdots & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

expand along 1st column

\vdots

$$= a_{11} a_{22} \cdots a_{nn}$$

Examples

1

$$\det \begin{bmatrix} 4 & 0 & 0 & 0 \\ 10 & \frac{1}{2} & 0 & 0 \\ 20 & -7 & 1 & 0 \\ -8 & 0 & 7 & 5 \end{bmatrix} =$$

2

$$\begin{vmatrix} 8 & 7 & 9 & -3 & 4 & 5 & 0 \\ 0 & 2 & 5 & 6 & 3 & 4 & -2 \\ 0 & 0 & 0 & 2 & 3 & -4 & -5 \\ 0 & 0 & 0 & 7 & 8 & -9 & 11 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{vmatrix} =$$

Weekend problem

Next time

For next time: Read Section PDM.

- Properties of determinants.
- Relation between determinants and invertibility of matrices.