MAT 1302B – Mathematical Methods II

Alistair Savage

Mathematics and Statistics
University of Ottawa

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Dimension

Vague definition: The dimension of a space is how many vectors you need to be able to get everywhere.

We want to make this more precise.

Linear independence: Precise definition of what it means to not have "more vectors than needed" to get around in a space.

Complementary question: We also need to "have enough" vectors to get everywhere – Span.

Subspaces: We restrict our attention to certain types of spaces – subspaces of $\mathbb{R}^n$.

Linear dependence

Definition

Consider an list of vectors $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ and the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}. \tag{1}$$

- If (1) has only the trivial solution, the set of vectors is linearly independent.
- If (1) has a nontrivial solution, the set of vectors is linearly dependent.

Put another way, the set of vectors is linearly dependent if there exist some scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}. \tag{2}$$

(2) is called a linear dependence relation.

Subspaces and bases

Definition (Subspace)

A subspace of $\mathbb{R}^n$ is any subset H of $\mathbb{R}^n$ that satisfies the following 3 conditions:

1. $\vec{0} \in H$.
2. If $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$.
3. If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$.

Definition (Basis)

Suppose H is a subspace of $\mathbb{R}^n$. A basis of H is a linearly independent set of vectors in H that span H.

In other words, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis of H if

- $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent, and
- $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = H$.

Important examples of subspaces

1. The set $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.
2. $\mathbb{R}^n$ is a subspace of itself.
3. Given vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, their span

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

   is a subspace of $\mathbb{R}^n$.
4. Column spaces: If A is an $m \times n$ matrix, then $\text{Col}\, A$ is the span of the columns of A and hence is a subspace of $\mathbb{R}^m$.

   $\text{Col}\, A$ is also the set of $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ has a solution.
5. Null spaces: If A is an $m \times n$ matrix, $\text{Nul}\, A$ is the set of solutions of the equation $A\vec{x} = \vec{0}$. That is,

$$\text{Nul}\, A = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

   $\text{Nul}\, A$ is a subspace of $\mathbb{R}^n$.

Basis of a null space

Question: Find a basis for the null space of

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}.$$

Solution: $\operatorname{Nul} A$ is the solution set of the equation $A\vec{x} = \vec{0}$. So we row reduce $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 9 & 5 & 4 & 0 \\ 1 & -1 & 6 & 5 & -3 & 0 \\ -2 & 0 & -6 & 1 & -2 & 0 \\ 4 & 1 & 9 & 1 & -9 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We return to equation form and write the general solution:

$$\begin{aligned} x_1 &= -3x_3 \\ x_2 &= 3x_3 + 7x_5 \\ x_4 &= 2x_5 \\ x_3, x_5 & \quad \text{free} \end{aligned}$$

Next, we write the solution set in vector parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_3, x_5 \in \mathbb{R}.$$

Therefore

$$\left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Nul} A$.

Why is it a basis of Nul A? We need to check the two conditions:

1. the set must span Nul A, and
2. the set must be linearly independent.

Since the general solution of $A\vec{x} = \vec{0}$ is

$$\left\{ x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; x_3, x_5 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\},$$

the set spans Nul A.

To check if the set is linearly independent, we note that if

$$x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

then $x_3 = x_5 = 0$ (look at the third and fifth components).

Basis of a null space

How to find a basis of the null space of a matrix

To find a basis of the null space of a matrix A, we:

1. Solve the homogeneous equation $A\vec{x} = \vec{0}$ by row reduction.
2. Write the general solution in vector parametric notation, as a span of vectors.
3. The set of vectors appearing in this form of the solution is a basis for $\operatorname{Nul} A$.

Thus, finding a basis of a null space involves just one extra step compared to problems we solved before (solving homogeneous equations).

Example

Find a basis for the solution set of the homogeneous system

$$\begin{array}{rcrcrcrcl} & & & & -x_3 & + & 3x_4 & = & 0 \\ -2x_1 & - & 4x_2 & - & 3x_3 & + & 9x_4 & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & - & 6x_4 & = & 0 \end{array}$$

Solution: This is the same as finding the null space of the coefficient matrix. We row reduce:

$$\left[\begin{array}{cccc|c} 0 & 0 & -1 & 3 & 0 \\ -2 & -4 & -3 & 9 & 0 \\ 1 & 2 & 2 & -6 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution in vector parametric form is:

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}$$

So a basis of the solution set is $\{(-2,1,0,0),(0,0,3,1)\}$.

Basis of a column space

Example: Find a basis for the column space of

$$B = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(Note that this is the RREF of the matrix in an earlier example.)

Solution: Label the columns: $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{bmatrix}$.
Note that the non-pivot columns are linear combinations of the pivot columns:

$$\vec{b}_3 = 3\vec{b}_1 - 3\vec{b}_2, \qquad \vec{b}_5 = -7\vec{b}_2 - 2\vec{b}_4.$$

Basis of a column space (cont.)

$$\vec{b}_3 = 3\vec{b}_1 - 3\vec{b}_2, \qquad \vec{b}_5 = -7\vec{b}_2 - 2\vec{b}_4$$

If $\vec{v}$ is in $\operatorname{Col} B = \operatorname{Span}\{\vec{b}_1, \dots, \vec{b}_5\}$, then

$$\begin{aligned}
\vec{v} &= c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 + c_5\vec{b}_5 \\
&= c_1\vec{b}_1 + c_2\vec{b}_2 + c_3(3\vec{b}_1 - 3\vec{b}_2) + c_4\vec{b}_4 + c_5(-7\vec{b}_2 - 2\vec{b}_4) \\
&= (c_1 + 3c_3)\vec{b}_1 + (c_2 - 3c_3 - 7c_5)\vec{b}_2 + (c_4 - 2c_5)\vec{b}_4
\end{aligned}$$

and so $\vec{v} \in \operatorname{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$. Therefore, $\operatorname{Col} B = \operatorname{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$.

Also, $\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$ is linearly independent since

$$\vec{0} = x_1\vec{b}_1 + x_2\vec{b}_2 + x_4\vec{b}_4 = x_1 \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + x_4 \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_4\\0\end{bmatrix} \implies x_1 = x_2 = x_4 = 0.$$

Thus, $\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$ is a basis for $\operatorname{Col} B$.

Basis of a column space (cont.)

Another example: Recall, from earlier example,

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \xrightarrow{\text{row reduce}} B = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let's find a basis for Col A.

Solution: Since A and B are row equivalent, the equations

$$A\vec{x} = \vec{0} \qquad \text{and} \qquad B\vec{x} = \vec{0}$$

have the same solution set.

Thus, the columns of A and B have the same linear dependence relations.

Basis of column space (cont.)

That is, if we label the columns:

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{bmatrix}, \quad B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{bmatrix}$$

then, since

$$\vec{b}_3 = 3\vec{b}_1 - 3\vec{b}_2, \qquad \vec{b}_5 = -7\vec{b}_2 - 2\vec{b}_4$$

we have

$$\vec{a}_3 = 3\vec{a}_1 - 3\vec{a}_2, \qquad \vec{a}_5 = -7\vec{a}_2 - 2\vec{a}_4 \qquad \text{(Check!)}$$

Thus, the pivot columns of A form a basis of $\operatorname{Col} A$. Since

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \xrightarrow{\text{row reduce}} B = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

a basis of $\operatorname{Col} A$ is

$$\{(1,1,-2,4), (-2,-1,0,1), (5,5,1,1)\}.$$

Finding the basis of a column space

Theorem

The pivot columns of a matrix form a basis for its column space.

Procedure for finding a basis of a column space

To find a basis for the column space of a matrix A we:

1. Row reduce the matrix to EF to determine which columns are the pivot columns.
2. A basis for the column space is the set of pivot columns of the original matrix A.

Notes:

- You must use the pivot columns of the original matrix and not the matrix you get after row reducing.
- This procedure allows us to find a basis of the span of any collection of vectors – we simply form a matrix with those vectors as columns.

Dimension

Theorem

Any two bases of a given subspace have the same number of vectors.

Definition (Dimension)

The dimension of a nonzero subspace H, denoted $\dim H$, is the number of vectors in any basis of H.

We define $\dim\{\vec{0}\} = 0$.

Example

Let

$$H = \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 12 \\ -2 \\ 11 \\ 7 \end{bmatrix} \right\}.$$

- Find a basis of H.
- What is the dimension of H?

Solution: We form a matrix with these vectors as columns and row reduce:

$$\begin{bmatrix} 3 & 6 & -6 & 3 & 9 & 12 \\ -1 & -2 & 3 & -2 & -6 & -2 \\ 2 & 4 & -3 & 2 & 6 & 11 \\ 1 & 2 & -1 & 1 & 3 & 7 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 2 & -2 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the first, third and fourth columns are the pivot columns.

Example (cont.)

Thus a basis of H is given by the first, third and fourth vectors in the original set:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

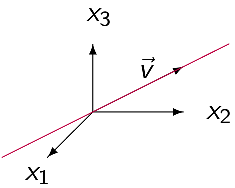
Since the basis has 3 elements, we have

$$\dim H = 3.$$

Examples

We can now confirm some of our intuition.

1. Consider a line through $\vec{0}$.



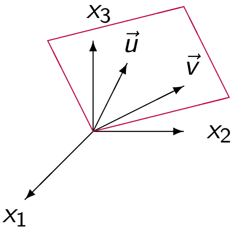
A basis for the line is $\{\vec{v}\}$ for any $\vec{v} \neq \vec{0}$ on the line. Why?

- The set $\{\vec{v}\}$ is linearly independent since $\vec{v} \neq \vec{0}$.
- Since $\text{Span}\{\vec{v}\} = \{t\vec{v} \mid t \in \mathbb{R}\}$ is the line, the set $\{\vec{v}\}$ spans the line.

Thus the line is 1-dimensional.

Examples (cont.)

2. Consider a plane through $\vec{0}$.



If we pick two non-parallel vectors $\vec{u}$ and $\vec{v}$ in the plane, then $\{\vec{u}, \vec{v}\}$ is a basis. Why?

- The set $\{\vec{u}, \vec{v}\}$ is linearly independent because $\vec{u}$ and $\vec{v}$ are not parallel.
- Since Span$\{\vec{u}, \vec{v}\}$ is the entire plane, the set $\{\vec{u}, \vec{v}\}$ spans the plane.

Thus the plane is 2-dimensional.

Examples (cont.)

3. Recall that $\mathbb{R}^n$ has the standard basis

$$\vec{e}_1 = (1, 0, \dots, 0), \quad \vec{e}_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad \vec{e}_n = (0, \dots, 0, 1).$$

Since this basis has n elements, $\mathbb{R}^n$ is n-dimensional.

Rank of a matrix

Definition (Rank)

The rank of a matrix A, denoted $\operatorname{rank} A$, is the dimension its column space.

Example: Find the rank of

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 5 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix}.$$

Solution: We row reduce:

$$A \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since A has 3 pivot columns, $\operatorname{Col} A$ has a basis with 3 vectors: $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$. Therefore, $\operatorname{rank} A = 3$.

Dimension of the null space

Question: What is the dimension of $\operatorname{Nul} A$?

Answer: Remember that we have

$$A \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the general solution to $A\vec{x} = \vec{0}$ will have 2 free variables, we have

$$\dim \operatorname{Nul} A = 2.$$

Rank and nullity of a matrix

Proposition

Suppose A is a matrix with n columns. Then

- $\operatorname{rank} A =$ number of pivot positions/columns in A,
- $\dim \operatorname{Nul} A = n -$ number of pivot positions in A.

The Rank Theorem

If a matrix A has n columns, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Terminology: $\dim \operatorname{Nul} A$ is sometimes called the nullity of A.

So to find $\operatorname{rank} A$ or $\dim \operatorname{Nul} A$ (the nullity of A), we row reduce to find out how many pivot positions A has.

Example

Find the rank and nullity of

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 & -1 & 2 & 4 \\ -1 & -2 & -3 & 3 & 1 & 2 & -7 \\ -3 & -6 & -12 & 0 & 2 & -5 & -10 \\ -5 & -10 & -20 & 1 & 3 & -3 & -9 \end{bmatrix}$$

Solution: We row reduce:

$$\begin{bmatrix} 1 & 2 & 4 & 1 & -1 & 2 & 4 \\ -1 & -2 & -3 & 3 & 1 & 2 & -7 \\ -3 & -6 & -12 & 0 & 2 & -5 & -10 \\ -5 & -10 & -20 & 1 & 3 & -3 & -9 \end{bmatrix} \xrightarrow{\text{RR}} \begin{bmatrix} 1 & 2 & 4 & 1 & -1 & 2 & 4 \\ 0 & 0 & 1 & -2 & 0 & 4 & -3 \\ 0 & 0 & 0 & 3 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 & 7 \end{bmatrix}.$$

Since A has 4 pivot positions, $\operatorname{rank} A = 4$, and the nullity of A is 3 (A has 3 columns that do not contain pivot positions).

Basis Theorem

The Basis Theorem

Suppose H is a p-dimensional subspace of $\mathbb{R}^n$. Then

1. Any linearly independent set of p vectors in H is a basis of H.
2. Any set of p vectors that span H is a basis of H.

Use of the theorem

- Suppose we know a subspace H has dimension p.
- We're given a set of p vectors in H.
- To check that the set is a basis of H, we only have to check linear independence or that the set spans H, not both.

Addition to Invertible Matrix Theorem

Recall from before that we had a list of conditions that were equivalent to a matrix being invertible.

We can now add some conditions to this list.

Addition to Invertible Matrix Theorem

Suppose A is an $n \times n$ matrix. Then the following statements are equivalent:

1. A is invertible.
2. The columns of A form a basis of $\mathbb{R}^n$.
3. $\operatorname{Col} A = \mathbb{R}^n$.
4. $\dim \operatorname{Col} A = n$.
5. $\operatorname{rank} A = n$.
6. $\operatorname{Nul} A = \{\vec{0}\}$.
7. $\dim \operatorname{Nul} A = 0$.

Next time

For next time: Read Section DM.

Determinants:

- Previously, we defined determinants for 2×2 matrices.
- Next time, we'll define determinants for matrices of arbitrary (square) size.
- Relation between determinants and invertibility.