MAT 1302B – Mathematical Methods II

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Announcements

Second Midterm:

- Next class.
- Covers the material from Lectures 1–11 (i.e. up to and including last lecture).
- Similar format to first midterm.
- Know your DGD number.
- No calculators allowed.
- Write in pen, not pencil.
- You may not leave in the final 10 minutes of the exam.

Overview

We want to develop rigorous definitions of intuitive concepts.

Motivating concept: Dimension

Sometimes our intuition about dimension breaks down:

- non-visual (or non-geometric) data (e.g. economic data)
- higher dimensions (> 3)
- space-filling curves
- fractals

Fractals (aside)

Fractals:

- exhibit self-similarity: small parts are a reduced-sized copy of the whole
- hard to describe using traditional geometry
- dimension is different depending of the definition of dimension that you use

Many natural objects approximate fractals:

- clouds
- mountain ranges
- lightning bolts
- coastlines
- snow flakes

Fractals (aside)

Mandelbrot set:

- example of a fractal
- easy to describe: a (complex) number c is in the Mandelbrot set if when you repeat the process "square the number and add c" over and over again, the results stay bounded (don't get infinitely large)
- exhibits extreme complexity

Video:

- Mandelbrot Set Zoom: http://youtu.be/G_GBwuYuOOs
- 3D version: `http://youtu.be/mMLOBkJltIw`

Mandelbrot applet:

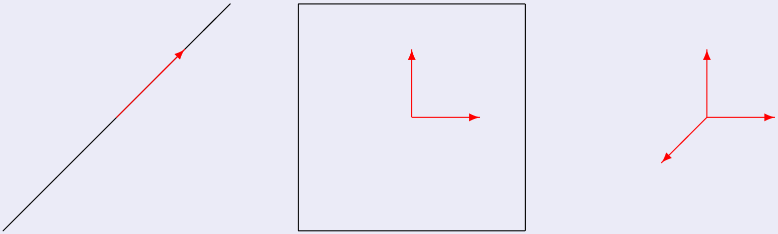
`http://math.hws.edu/xJava/MB/`

Motivation: dimension

What is dimension?

Intuitive idea: Dimension is how many directions we need to “get everywhere” in a space.

Line | Plane | Space

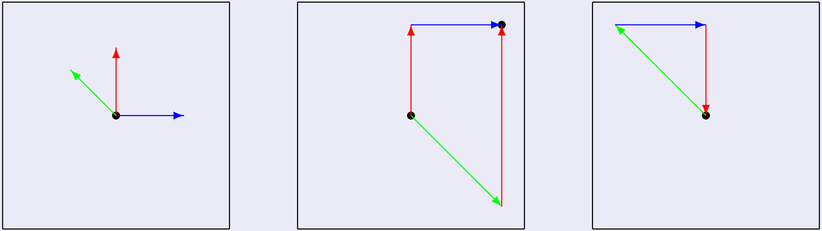


One dimensional | Two dimensional | Three dimensional

Motivation: dimension

Problem

What about having three directions in a plane?



Resolution

- One direction is not needed (is redundant).
- With the three vectors, there are multiple ways to get to the same place (even if we use a multiple of each vector only once).
- Alternatively: We can return to the origin, using (a multiple of) each vector only once.

Linear dependence

Definition

Consider a list of vectors $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ and the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}. \tag{1}$$

- If (1) has only the trivial solution, the vectors are linearly independent.
- If (1) has a nontrivial solution, the vectors are linearly dependent.

Put another way, the vectors are linearly dependent if there exist some scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}. \tag{2}$$

(2) is called a linear dependence relation.

Dimension

Vague definition: The dimension of a space is how many vectors you need to be able to get everywhere.

We want to make this more precise.

Linear independence: Precise definition of what it means to not have "more vectors than needed" to get around in a space.

Complementary question: We also need to "have enough" vectors to get everywhere.

We concentrate on making this concept precise today.

First step: We need to restrict our attention to certain types of spaces – subspaces of $\mathbb{R}^n$.

Subspaces of $\mathbb{R}^n$

Definition

A subspace of $\mathbb{R}^n$ is any subset H of $\mathbb{R}^n$ that satisfies:

1. $\vec{0} \in H$.
2. If $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$. (H is closed under vector addition.)
3. If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$. (H is closed under scalar multiplication.)

Example 1

$H = \{\vec{0}\}$. Is H a subspace of $\mathbb{R}^n$? We check the 3 conditions:

1. $\vec{0} \in H$ ✓
2. If $\vec{u}, \vec{v} \in H$, then $\vec{u} = \vec{v} = \vec{0}$.
   Thus $\vec{u} + \vec{v} = \vec{0} \in H$. ✓
3. If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $\vec{u} = \vec{0}$.
   Thus $c\vec{u} = c\vec{0} = \vec{0} \in H$. ✓

Thus H is a subspace of $\mathbb{R}^n$.

Subspaces of $\mathbb{R}^n$

Example 2

Is $U = \{(1,0)\}$ a subspace of $\mathbb{R}^2$?

1. $\vec{0} \notin U$. $\times$

Thus U is not a subspace of $\mathbb{R}^2$ (it fails Condition 1)

Example 3

Is $V = \{(0,0),(1,0)\}$ a subspace of $\mathbb{R}^2$?

1. $\vec{0} \in V$. $\checkmark$
2. $(1,0) + (1,0) = (2,0) \notin V$. $\times$
3. $3(1,0) = (3,0) \notin V$. $\times$

Thus V is not a subspace of $\mathbb{R}^2$ (it fails Conditions 2 and 3).

Subspaces of $\mathbb{R}^n$

Example 4

Is $H = \mathbb{R}^n$ a subspace of $\mathbb{R}^n$?

1. $\vec{0} \in H$. ✓
2. $\vec{u}, \vec{v} \in H \implies \vec{u} + \vec{v} \in H$. ✓
   (Since the sum of any two vectors in $\mathbb{R}^n$ is another vector in $\mathbb{R}^n$.)
3. $\vec{u} \in H,\ c \in \mathbb{R} \implies c\vec{u} \in H$. ✓
   (Since any multiple of any vector in $\mathbb{R}^n$ is another vector in $\mathbb{R}^n$.)

So $\mathbb{R}^n$ is a subspace of itself.

Important example: Spans

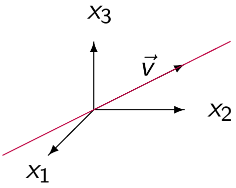
Example: If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are in $\mathbb{R}^n$, then $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$.

Justification: We check the 3 conditions:

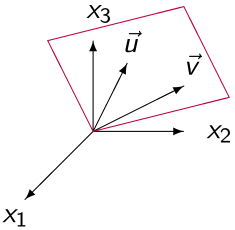
1. $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ ✓
2. Suppose $\vec{u}, \vec{v} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$. Then
$$\vec{u} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \text{ for some } c_1, \dots, c_k \in \mathbb{R}, \quad \text{and}$$
$$\vec{v} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k \text{ for some } d_1, \dots, d_k \in \mathbb{R}.$$
Then
$$\vec{u} + \vec{v} = (c_1 + d_1)\vec{v}_1 + \cdots + (c_k + d_k)\vec{v}_k \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}. \qquad \checkmark$$
3. Suppose $\vec{u} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ and $c \in \mathbb{R}$. Then
$$\vec{u} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k \text{ for some } a_1, \dots, a_k \in \mathbb{R}.$$
Thus
$$c\vec{u} = (ca_1)\vec{v}_1 + \cdots + (ca_k)\vec{v}_k \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}. \qquad \checkmark$$

Corollary

1. Lines through the origin are subspaces since they are of the form $\text{Span}\{\vec{v}\}$ for some nonzero $\vec{v}$ on the line.



2. Planes through the origin are subspaces since they are of the form $\text{Span}\{\vec{u}, \vec{v}\}$.



Spans and column spaces

Note: Lines or planes not containing the origin are not subspaces.

Why not? They violate Condition 1 (they do not contain $\vec{0}$).

Definition

The column space of a matrix A is the set of all linear combinations of the columns of A (i.e. the span of the columns of A).

It is denoted $\operatorname{Col} A$.

Since spans are subspaces, the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

Example

If

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 5 \\ -2 & 4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},$$

is $\vec{b} \in \operatorname{Col} A$?

Solution: Remember that $\vec{b}$ is a linear combination of the columns of A if

$$x_1 \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} = \vec{b}$$

has a solution, i.e. if $A\vec{x} = \vec{b}$ has a solution.

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ -4 & 7 & 5 & 2 \\ -2 & 4 & 6 & 4 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solution. So answer is no, $\vec{b} \notin \operatorname{Col} A$.

Remarks

1. Another way of asking the same question:

$$\text{Is } \vec{b} \in \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \right\} ?$$

2. We see from the above that Col A is the set of all $\vec{b}$ for which

$$A\vec{x} = \vec{b}$$

has a solution.

So column spaces are closely related to linear systems!

Null space

Definition

The null space of a matrix A, denoted $\operatorname{Nul} A$, is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$.

$$\operatorname{Nul} A = \{\vec{x} \mid A\vec{x} = \vec{0}\}$$

Null spaces are subspaces

Theorem

If A is an $m \times n$ matrix, then $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

Equivalently, the set of all solutions to the system $A\vec{x} = \vec{0}$ of m homogeneous equations in n unknowns is a subspace of $\mathbb{R}^n$.

Justification of theorem

We need to check the 3 conditions:

1. $A\vec{0} = \vec{0} \implies \vec{0} \in \operatorname{Nul} A$ ✓
2. Suppose $\vec{u}, \vec{v} \in \operatorname{Nul} A$.
   Then $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$.
   So $\vec{u} + \vec{v} \in \operatorname{Nul} A$. ✓
3. Suppose $\vec{u} \in \operatorname{Nul} A$ and $c \in \mathbb{R}$.
   Then $A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$.
   So $c\vec{u} \in \operatorname{Nul} A$. ✓

So $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

Bases

Now we put the concepts

- linear independence ("not too many"), and
- span ("enough")

together.

Definition (Basis)

Suppose H is a subspace of $\mathbb{R}^n$. A basis of H is a linearly independent set of vectors in H that span H.

In other words, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis of H if

- $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, and
- $\mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = H$.

Notes:

- A basis of a subspace is a "minimal" spanning set.
- The plural of 'basis' is 'bases'.

Example

The vectors

$$\vec{e}_1 = (1, 0, \ldots, 0), \quad \vec{e}_2 = (0, 1, 0, \ldots, 0), \ldots \quad , \quad \vec{e}_n = (0, \ldots, 0, 1)$$

form a basis of $\mathbb{R}^n$ (called the standard basis).

- There are linearly independent since

$$x_1\vec{e}_1 + \cdots + x_n\vec{e}_n = \vec{0} \implies (x_1, \ldots, x_n) = \vec{0} \implies x_1 = 0, \ldots, x_n = 0.$$

- They span $\mathbb{R}^n$ since given any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

$$(x_1, \ldots, x_n) = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n \implies (x_1, \ldots, x_n) \in \text{Span}\{\vec{e}_1, \ldots, \vec{e}_n\}.$$

Note: The vectors $\vec{e}_1, \ldots, \vec{e}_n$ are the columns of I_n.

Theorem

The columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$.

Examples

Which of the following are subspaces of $\mathbb{R}^n$ for the given n?

1. $\{(0,0,0)\}$, $n = 2$. NO

2. $\{(0,0,0)\}$, $n = 3$. YES

3. $\{(1,0,0)\}$, $n = 3$. NO

4. $H = \{(a,0,0) \mid a \in \mathbb{R}\}$, $n = 3$
   - $a = 0 \implies (0,0,0) \in H$ ✓
   - $(a_1,0,0) + (a_2,0,0) = (a_1 + a_2, 0, 0) \in H$ ✓
   - $c(a,0,0) = (ca,0,0) \in H$ ✓

   So YES, H is a subspace of $\mathbb{R}^3$. You can also see this by noting that $H = \text{Span}\{(1,0,0)\}$.

5. $U = \{(x,x^2) \mid x \in \mathbb{R}\}$, $n = 2$.

   $(1,1) \in U$, but $2(1,1) = (2,2) \notin U$.

   So NO, U is not a subspace of $\mathbb{R}^2$.

Examples (cont.)

6. $V = \{(x^3, 0) \mid x \in \mathbb{R}\}$, $n = 2$.

   Note that any real number is the cube of some other real number. Thus
   $$V = \{(y, 0) \mid y \in \mathbb{R}\}.$$
   Then
   - $(0, 0) \in V$ ✓
   - $(a, 0) + (b, 0) = (a + b, 0) \in V$ ✓
   - $c(a, 0) = (ca, 0) \in V$ ✓

   So, YES, V is a subspace of $\mathbb{R}^2$. You can also see this by noting that $V = \text{Span}\{(1, 0)\}$.

7. $\{(x, y, z) \mid 2x + y + 3z = 0\}$, $n = 3$.

   This is the solution set for a homogeneous system, so YES, it is a subspace.

8. $\{(x, y, z) \mid 2x + y + 3z = 1\}$, $n = 3$.

   NO, since the set does not contain $(0, 0, 0)$.

Examples (cont.)

9. The set of all vectors of the form

$$\begin{bmatrix} 2a - 3b \\ 3a + b \\ -2 \end{bmatrix}$$

for a and b arbitrary real numbers ($n = 3$).

NO, since this set does not include the zero vector.

10. The set

$$\{(1 - c, 3 + c) \mid c \in \mathbb{R}\}, \qquad n = 2.$$

NO, since this set does not include the zero vector.

In order for the first coordinate to be zero, we need $c = 1$. But in order for the second coordinate to be zero, we need $c = -3$. Therefore, this is no single value of c for which both coordinates are zero.

Examples (cont.)

11. Consider the set of all vectors of the form

$$(3a+6b-c, -2a+7c, c-3b, 0)$$

for a, b, c arbitrary real numbers ($n=4$).

Since

$$\begin{bmatrix} 3a+6b-c \\ -2a+7c \\ c-3b \\ 0 \end{bmatrix} = a\begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 6 \\ 0 \\ -3 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 7 \\ 1 \\ 0 \end{bmatrix}$$

we see that this set is

$$\operatorname{Span}\left\{ \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and thus is a subspace of $\mathbb{R}^4$. So the answer is YES.

Examples (cont.)

12. Let

$$\vec{v}_1 = (2,1,1), \quad \vec{v}_2 = (4,3,2), \quad \vec{v}_3 = (-2,0,0), \quad \vec{w} = (8,4,4)$$

(a) Is $\vec{w}$ in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? How many vectors are in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
(b) How many vectors are in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
(c) Is $\vec{w}$ in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

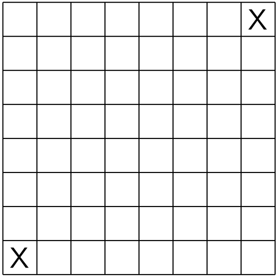
Solution:

(a) NO, since $\vec{w}$ is not equal to $\vec{v}_1$, $\vec{v}_2$ or $\vec{v}_3$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ has 3 vectors.
(b) $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ has an infinite number of vectors.
(c) To determine this, we row reduce

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 8 \\ 1 & 3 & 0 & 4 \\ 1 & 2 & 0 & 4 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The corresponding system is consistent. Thus the answer is YES.

Weekend problem – last time



- Suppose you have an 8×8 checkerboard.
- Remove two diagonally opposite corners.
- Place dominoes on the board so that they cover two neighbouring (vertically or horizontally) squares. Dominoes cannot hang over the edge of the board.

Question: Can you cover every square without the dominoes overlapping?

Weekend problem – solution

Answer: No!

Why not?

- Remember that the squares of a checkerboard alternate colour.
- When you remove two diagonally opposite corners, you remove squares of the same colour.
- So of the remaining squares, there are 2 more of one color than the other.
- Each domino covers one square of each color.
- So it's impossible to cover the board with dominos.

Next time

For next time: Read Sections D, PD, HSE.NSM

- How to find a basis for a give subspace.
- Dimension (precise definition).
- Relation between dimension and matrices (rank, column space, null space).