

MAT 1302B – Mathematical Methods II

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Announcements

Second Midterm:

- Next class.
- Covers the material from Lectures 1–11 (i.e. up to and including last lecture).
- Similar format to first midterm.
- Know your DGD number.
- No calculators allowed.
- Write in **pen**, not pencil.
- You may not leave in the final 10 minutes of the exam.

Overview

We want to develop rigorous definitions of intuitive concepts.

Motivating concept: Dimension

Sometimes our intuition about dimension breaks down:

- non-visual (or non-geometric) data (e.g. economic data)
- higher dimensions (> 3)
- space-filling curves
- fractals

Fractals (aside)

Fractals:

- exhibit **self-similarity**: small parts are a reduced-sized copy of the whole
- hard to describe using traditional geometry
- dimension is different depending of the definition of dimension that you use

Many natural objects approximate fractals:

- clouds
- mountain ranges
- lightning bolts
- coastlines
- snow flakes

Fractals (aside)

Mandelbrot set:

- example of a fractal
- easy to describe: a (complex) number c is in the Mandelbrot set if when you repeat the process “square the number and add c ” over and over again, the results stay bounded (don’t get infinitely large)
- exhibits extreme complexity

Video:

- **Mandelbrot Set Zoom:** http://youtu.be/G_GBwuYuOOs
- **3D version:** <http://youtu.be/mMLOBkJltIw>

Mandelbrot applet:

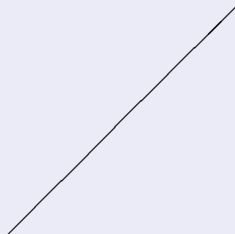
<http://math.hws.edu/xJava/MB/>

Motivation: dimension

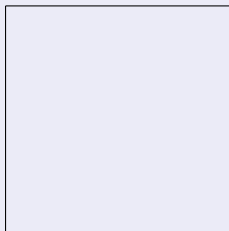
What is dimension?

Intuitive idea:

Line



Plane

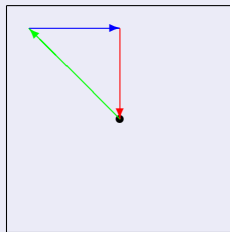
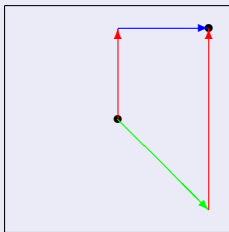
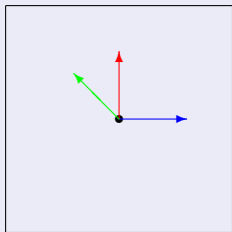


Space

Motivation: dimension

Problem

What about having three directions in a plane?



Resolution

- One direction is not needed (is redundant).
- With the three vectors, there are multiple ways to get to the same place (even if we use a multiple of each vector only once).
- **Alternatively:** We can return to the origin, using (a multiple of) each vector only once.

Linear dependence

Definition

Consider a list of vectors $\vec{v}_1, \dots, \vec{v}_p$ in \mathbb{R}^n and the equation

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_p \vec{v}_p = \vec{0}. \quad (1)$$

- If (1) has only the trivial solution, the vectors are **linearly independent**.
- If (1) has a nontrivial solution, the vectors are **linearly dependent**.

Put another way, the vectors are linearly dependent if there exist some scalars c_1, \dots, c_p , not all zero, such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p = \vec{0}. \quad (2)$$

(2) is called a **linear dependence relation**.

Dimension

Vague definition: The dimension of a space is how many vectors you need to be able to get everywhere.

We want to make this more precise.

Linear independence: Precise definition of what it means to not have “more vectors than needed” to get around in a space.

Complementary question: We also need to “have enough” vectors to get everywhere.

We concentrate on making this concept precise today.

First step: We need to restrict our attention to certain types of spaces – subspaces of \mathbb{R}^n .

Subspaces of \mathbb{R}^n

Definition

A **subspace** of \mathbb{R}^n is any subset H of \mathbb{R}^n that satisfies:

- 1 $\vec{0} \in H$.
- 2 If $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$. (H is **closed under vector addition**.)
- 3 If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$. (H is **closed under scalar multiplication**.)

Example 1

$H = \{\vec{0}\}$. Is H a subspace of \mathbb{R}^n ?

Subspaces of \mathbb{R}^n

Example 2

Is $U = \{(1, 0)\}$ a subspace of \mathbb{R}^2 ?

Example 3

Is $V = \{(0, 0), (1, 0)\}$ a subspace of \mathbb{R}^2 ?

Subspaces of \mathbb{R}^n

Example 4

Is $H = \mathbb{R}^n$ a subspace of \mathbb{R}^n ?

Important example: Spans

Example: If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are in \mathbb{R}^n , then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of \mathbb{R}^n .

Justification: We check the 3 conditions:

① $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_k \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ ✓

② Suppose $\vec{u}, \vec{v} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$. Then

$$\vec{u} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k \text{ for some } c_1, \dots, c_k \in \mathbb{R}, \quad \text{and}$$

$$\vec{v} = d_1\vec{v}_1 + \dots + d_k\vec{v}_k \text{ for some } d_1, \dots, d_k \in \mathbb{R}.$$

Then

$$\vec{u} + \vec{v} = (c_1 + d_1)\vec{v}_1 + \dots + (c_k + d_k)\vec{v}_k \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}. \quad \checkmark$$

③ Suppose $\vec{u} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ and $c \in \mathbb{R}$. Then

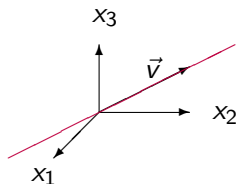
$$\vec{u} = a_1\vec{v}_1 + \dots + a_k\vec{v}_k \text{ for some } a_1, \dots, a_k \in \mathbb{R}.$$

Thus

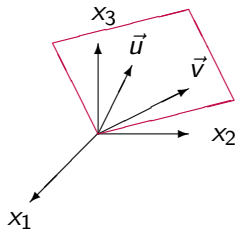
$$c\vec{u} = (ca_1)\vec{v}_1 + \dots + (ca_k)\vec{v}_k \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}. \quad \checkmark$$

Corollary

- 1 Lines through the origin are subspaces since they are of the form $\text{Span}\{\vec{v}\}$ for some nonzero \vec{v} on the line.



- 2 Planes through the origin are subspaces since they are of the form $\text{Span}\{\vec{u}, \vec{v}\}$.



Spans and column spaces

Note: Lines or planes **not** containing the origin are **not** subspaces.

Why not?

Definition

The **column space** of a matrix A is the set of all linear combinations of the columns of A (i.e. the span of the columns of A).

It is denoted $\text{Col } A$.

Since spans are subspaces, the column space of an $m \times n$ matrix is a subspace of \mathbb{R}^m .

Example

If

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 7 & 5 \\ -2 & 4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},$$

is $\vec{b} \in \text{Col } A$?

Solution:

Remarks

① Another way of asking the same question:

② We see from the above that $\text{Col } A$ is the set of all \vec{b} for which

$$A\vec{x} = \vec{b}$$

has a solution.

So column spans are closely related to linear systems!

Null space

Definition

The **null space** of a matrix A , denoted $\text{Nul } A$, is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$.

$$\text{Nul } A = \{\vec{x} \mid A\vec{x} = \vec{0}\}$$

Null spaces are subspaces

Theorem

If A is an $m \times n$ matrix, then $\text{Nul } A$ is a subspace of \mathbb{R}^n .

Equivalently, the set of all solutions to the system $A\vec{x} = \vec{0}$ of m homogeneous equations in n unknowns is a subspace of \mathbb{R}^n .

Justification of theorem

We need to check the 3 conditions:

- 1 $A\vec{0} = \vec{0} \implies \vec{0} \in \text{Nul } A \quad \checkmark$
- 2 Suppose $\vec{u}, \vec{v} \in \text{Nul } A$.
Then $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$.
So $\vec{u} + \vec{v} \in \text{Nul } A. \quad \checkmark$
- 3 Suppose $\vec{u} \in \text{Nul } A$ and $c \in \mathbb{R}$.
Then $A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$.
So $c\vec{u} \in \text{Nul } A. \quad \checkmark$

So $\text{Nul } A$ is a subspace of \mathbb{R}^n .

Bases

Now we put the concepts

- linear independence (“not too many”), and
- span (“enough”)

together.

Definition (Basis)

Suppose H is a subspace of \mathbb{R}^n . A **basis** of H is a linearly independent set of vectors in H that span H .

In other words, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis of H if

- $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, and
- $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = H$.

Notes:

- A basis of a subspace is a “minimal” spanning set.
- The plural of ‘basis’ is ‘bases’.

Example

The vectors

$$\vec{e}_1 = (1, 0, \dots, 0), \quad \vec{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \vec{e}_n = (0, \dots, 0, 1)$$

form a basis of \mathbb{R}^n (called the **standard basis**).

- There are linearly independent since

$$x_1 \vec{e}_1 + \dots + x_n \vec{e}_n = \vec{0} \implies (x_1, \dots, x_n) = \vec{0} \implies x_1 = 0, \dots, x_n = 0.$$

- They span \mathbb{R}^n since given any vector $(x_1, \dots, x_n) \in \mathbb{R}^n$, we have

$$(x_1, \dots, x_n) = x_1 \vec{e}_1 + \dots + x_n \vec{e}_n \implies (x_1, \dots, x_n) \in \text{Span}\{\vec{e}_1, \dots, \vec{e}_n\}.$$

Note: The vectors $\vec{e}_1, \dots, \vec{e}_n$ are the columns of I_n .

Theorem

The columns of an invertible $n \times n$ matrix form a basis of \mathbb{R}^n .

Examples

Which of the following are subspaces of \mathbb{R}^n for the given n ?

- 1 $\{(0, 0, 0)\}$, $n = 2$.
- 2 $\{(0, 0, 0)\}$, $n = 3$.
- 3 $\{(1, 0, 0)\}$, $n = 3$.
- 4 $H = \{(a, 0, 0) \mid a \in \mathbb{R}\}$, $n = 3$
- 5 $U = \{(x, x^2) \mid x \in \mathbb{R}\}$, $n = 2$.

Examples (cont.)

6 $V = \{(x^3, 0) \mid x \in \mathbb{R}\}, n = 2.$

7 $\{(x, y, z) \mid 2x + y + 3z = 0\}, n = 3.$

8 $\{(x, y, z) \mid 2x + y + 3z = 1\}, n = 3.$

Examples (cont.)

- 9 The set of all vectors of the form

$$\begin{bmatrix} 2a - 3b \\ 3a + b \\ -2 \end{bmatrix}$$

for a and b arbitrary real numbers ($n = 3$).

- 10 The set

$$\{(1 - c, 3 + c) \mid c \in \mathbb{R}\}, \quad n = 2.$$

Examples (cont.)

- 11 Consider the set of all vectors of the form

$$(3a + 6b - c, -2a + 7c, c - 3b, 0)$$

for a, b, c arbitrary real numbers ($n = 4$).

Examples (cont.)

12 Let

$$\vec{v}_1 = (2, 1, 1), \quad \vec{v}_2 = (4, 3, 2), \quad \vec{v}_3 = (-2, 0, 0), \quad \vec{w} = (8, 4, 4)$$

- (a) Is \vec{w} in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? How many vectors are in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
- (b) How many vectors are in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
- (c) Is \vec{w} in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

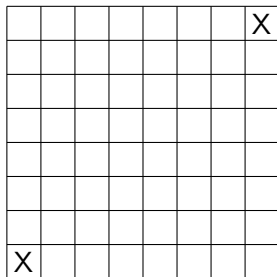
Solution:

(a)

(b)

(c)

Weekend problem – last time



- Suppose you have an 8×8 checkerboard.
- Remove two diagonally opposite corners.
- Place dominoes on the board so that they cover two neighbouring (vertically or horizontally) squares. Dominoes cannot hang over the edge of the board.

Question: Can you cover every square without the dominoes overlapping?

Weekend problem – solution

Next time

For next time: Read Sections D, PD, HSE.NSM

- How to find a basis for a give subspace.
- Dimension (precise definition).
- Relation between dimension and matrices (rank, column space, null space).