MAT 1302B – Mathematical Methods II

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Announcements

Second Midterm:

- Friday, March 6.
- Covers the material from Lectures 1–11 (i.e. up to and including today's lecture).

Recap – matrix operations

- matrix multiplication
  - sizes: $(m \times n)(n \times k) = (m \times k)$
  - Example:

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ 2 & 11 \end{bmatrix}$$

- matrix transpose
  - Example:

$$\begin{bmatrix} 3 & 4 & 0 & -2 \\ 10 & -7 & 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 10 \\ 4 & -7 \\ 0 & 3 \\ -2 & 4 \end{bmatrix}$$

  - $(AB)^T = B^T A^T$
- matrix inverses

Last time – matrix inverses

Definition (Inverse of a matrix)

An $n \times n$ (square) matrix A is invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n$$

We call C an inverse of A. It is unique and we denote it A^{-1}.

Theorem (Inverses of 2×2 matrices)

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If $\det A = ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $\det A = ad - bc = 0$, then A is not invertible.

Last time – matrix inverses

- To find the inverse of A, row reduce the superaugmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.
- When row reducing, the operations we perform are determined by the left side of the superaugmented matrix (the right side "just comes along for the ride").
- If you get a row of zeros in the left matrix (left of the vertical bar), stop. The matrix A is not invertible.
- Otherwise, keep going until you get I on the left.
- Then the matrix on the right is A^{-1}.
- Check your answer by verifying that $AA^{-1} = I$ or $A^{-1}A = I$.

Example

Is the following matrix invertible? If so, find its inverse.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 4 & 6 & 8 \\ -2 & 2 & 4 \end{bmatrix}$$

Solution: We row reduce the superaugmented matrix.

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ 4 & 6 & 8 & 0 & 1 & 0 \\ -2 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_1 + R_3]{-R_1 + R_2} \left[\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & 6 & -1 & 1 & 0 \\ 0 & 3 & 5 & \frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{3}{4}R_2 + R_3} \left[\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & 6 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{-3}{4} & 1 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & 6 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & 2 \end{array}\right]$$

Example (cont.)

$$\xrightarrow[-2R_3+R_1]{-6R_3+R_2} \left[\begin{array}{ccc|ccc} 4 & 2 & 0 & -4 & 3 & -4 \\ 0 & 4 & 0 & -16 & 10 & -12 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & 2 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 4 & 2 & 0 & -4 & 3 & -4 \\ 0 & 1 & 0 & -4 & \frac{5}{2} & -3 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & 2 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 4 & -2 & 2 \\ 0 & 1 & 0 & -4 & \frac{5}{2} & -3 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & 2 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & \frac{5}{2} & -3 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & 2 \end{array}\right]$$

So A is invertible (we obtained I on the left hand side) and

$$A^{-1} = \begin{bmatrix} 1 & -1/2 & 1/2 \\ -4 & 5/2 & -3 \\ 5/2 & -3/2 & 2 \end{bmatrix}$$

Check your answer!

$$AA^{-1} = \begin{bmatrix} 4 & 2 & 2 \\ 4 & 6 & 8 \\ -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 1/2 \\ -4 & 5/2 & -3 \\ 5/2 & -3/2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

Invertible Matrix Theorem

If A is an $n \times n$ square matrix, the following statements are equivalent:

1. A is an invertible matrix.
2. A is row equivalent to I_n.
3. A has n pivot positions.
4. The equation $A\vec{x} = \vec{0}$ has only the trivial solution.
5. The columns of A are linearly independent.
6. The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$.
7. The span of the columns of A is all of $\mathbb{R}^n$.
8. There is an $n \times n$ matrix C such that $CA = I_n$.
9. There is an $n \times n$ matrix D such that $AD = I_n$.
10. A^T is invertible.

Warning

The Invertible Matrix Theorem assumes the matrix is square!

Example: If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

then $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$ but A is not invertible!

Note

If A and B are square and $AB = I$, then both A and B are invertible and

$$A^{-1} = B \qquad \text{and} \qquad B^{-1} = A$$

We don't need to check that $BA = I$.

Leontief Input-Output Model

Leontief Input-Output Model: An economic model describing how changes in one economic sector affect other sectors.

Wassily Leontief (1905-1999)

- Born in Germany, raised in Russia.
- Became professor of economics at Harvard.
- Used computers at Harvard to model economic data.
- Modeled each economic sector with a linear equation and used computer to solve the system.
- One of the first significant uses of computers for mathematical modeling.

In 1973, Leontief won the Nobel Prize in Economics for his work.

Leontief Input-Output Model

Suppose the economy is divided into n sectors (manufacturing, agriculture, services, etc.). We have

- production vector

$$\vec{x} = \begin{bmatrix} \text{units produced by sector 1} \\ \text{units produced by sector 2} \\ \vdots \\ \text{units produced by sector n} \end{bmatrix} \in \mathbb{R}^n$$

- final demand vector

$$\vec{d} = \begin{bmatrix} \text{units consumed from sector 1} \\ \text{units consumed from sector 2} \\ \vdots \\ \text{units consumed from sector n} \end{bmatrix} \in \mathbb{R}^n$$

(goods consumed by non-producing part of the economy)

Intermediate demand: In order for each sector to produce goods, it consumes goods from the other sectors (and its own sector).

We want:

$$\underbrace{\vec{x}}_{\text{amount produced}} \quad = \quad \text{intermediate demand} \quad + \quad \underbrace{\vec{d}}_{\text{final demand}}$$

For each sector S we have a unit consumption vector in $\mathbb{R}^n$ that tells us the inputs needed from each sector per unit of output of the sector S.

Example: In order for the manufacturing sector to produce one unit of output, it must consume:

- 0.20 units from the manufacturing sector (its own sector),
- 0.20 units from the agriculture sector,
- 0.10 units from the services sector.

Example

Suppose the economy is divided into three sectors: manufacturing, agriculture, and services.

Inputs consumed per unit of output:

Purchased From	Manufacturing	Agriculture	Services
Manufacturing	.20	.20	.30
Agriculture	.20	.50	.10
Services	.10	.10	.30
	$\uparrow$	$\uparrow$	$\uparrow$
	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$

$\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$ are the unit consumption vectors.

Example (cont.)

Purchased From	**Manufacturing**	**Agriculture**	**Services**
Manufacturing	.20	.20	.30
Agriculture	.20	.50	.10
Services	.10	.10	.30
	$\uparrow$	$\uparrow$	$\uparrow$
	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$

Intermediate demands: If agriculture decides to produce 200 units of output, its intermediate demands are given by

$$200\vec{c}_2 = 200 \begin{bmatrix} .2 \\ .5 \\ .1 \end{bmatrix} = \begin{bmatrix} 40 \\ 100 \\ 20 \end{bmatrix} \begin{matrix} \leftarrow \text{40 units from manufacturing} \\ \leftarrow \text{100 units from agriculture} \\ \leftarrow \text{20 units from services} \end{matrix}$$

Example (cont.)

So if

- manufacturing wants to produce x_1 units,
- agriculture wants to produce x_2 units, and
- services wants to produce x_3 units,

then the total intermediate demand is

$$x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3 = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C\vec{x}$$

where $C = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_2 \end{bmatrix}$ is the consumption matrix.

Leontief Input-Output Model (Production Equation)

$$\underbrace{\vec{x}}_{\text{amount produced}} = \underbrace{C\vec{x}}_{\text{intermediate demand}} + \underbrace{\vec{d}}_{\text{final demand}}$$

We can rewrite this equation as

$$\begin{aligned} I\vec{x} &= C\vec{x} + \vec{d} \\ \Leftrightarrow I\vec{x} - C\vec{x} &= \vec{d} \\ \Leftrightarrow (I - C)\vec{x} &= \vec{d} \qquad \leftarrow \text{equation we solve} \end{aligned}$$

So to solve a problem involving the Leontief Input-Output Model, we

1. use the information given in the problem to write down the consumption matrix C and final demand vector $\vec{d}$, and
2. solve the matrix equation $(I - C)\vec{x} = \vec{d}$ for the production vector $\vec{x}$.

This tells us the amount that each sector must produce to meet the final demand.

Example

Consider our previous example:

Purchased From	**Manufacturing**	**Agriculture**	**Services**
Manufacturing	.20	.20	.30
Agriculture	.20	.50	.10
Services	.10	.10	.30
	$\uparrow$	$\uparrow$	$\uparrow$
	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$

Find the production levels that satisfy a final demand of 152 units from manufacturing, 2 units from agriculture and 19 units from services.

Solution: We want to solve $(I - C)\vec{x} = \vec{d}$. We have

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \underbrace{\begin{bmatrix} .2 & .2 & .3 \\ .2 & .5 & .1 \\ .1 & .1 & .3 \end{bmatrix}}_{\text{consumption matrix}} = \begin{bmatrix} .8 & -.2 & -.3 \\ -.2 & .5 & -.1 \\ -.1 & -.1 & .7 \end{bmatrix}, \quad \underbrace{\vec{d} = \begin{bmatrix} 152 \\ 2 \\ 19 \end{bmatrix}}_{\text{final demand vector}}.$$

Example (cont.)

To solve $(I - C)\vec{x} = \vec{d}$, we row reduce

$$\left[\begin{array}{c|c} I - C & \vec{d} \end{array}\right] = \left[\begin{array}{ccc|c} .8 & -.2 & -.3 & 152 \\ -.2 & .5 & -.1 & 2 \\ -.1 & -.1 & .7 & 19 \end{array}\right]$$

$$\xrightarrow[10R_3]{\substack{10R_1 \\ 10R_2}} \left[\begin{array}{ccc|c} 8 & -2 & -3 & 1520 \\ -2 & 5 & -1 & 20 \\ -1 & -1 & 7 & 190 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 250 \\ 0 & 1 & 0 & 120 \\ 0 & 0 & 1 & 80 \end{array}\right]$$

Thus

- manufacturing must produce 250 units,
- agriculture must produce 120 units, and
- services must produce 80 units.

Notes

1. In order for the production to be feasible, all entries of the production vector $\vec{x}$ should be nonnegative.

2. If $(I - C)$ is invertible, we can solve the problem using matrix inverses:

$$(I - C)\vec{x} = \vec{d} \quad \Longrightarrow \quad \vec{x} = (I - C)^{-1}\vec{d}$$

   Once we compute $(I - C)^{-1}$, we can easily find the production levels needed for different final demands.

Question: Are there some conditions that would guarantee that

- $(I - C)$ is invertible, and
- the production vector has nonnegative entries?

Answer: Yes!

Definition

A column sum is the sum of all the entries in a certain column of a matrix.

Theorem

Suppose C is the consumption matrix and $\vec{d}$ is the final demand vector. If

1. C and $\vec{d}$ have nonnegative entries, and
2. each column sum of C is less than 1,

then $(I - C)$ is invertible and the production vector

$$\vec{x} = (I - C)^{-1}\vec{d}$$

has nonnegative entries and is the unique solution of the production equation

$$\vec{x} = C\vec{x} + \vec{d} \qquad (\text{ or } (I - C)\vec{x} = \vec{d}\ ).$$

Example

Economy consists of 3 sectors: technology, transportation, and mining.

- Each unit produced by the tech sector requires .6 units from the tech sector, .1 from transportation and .2 from mining.
- Each unit produced by transportation requires 0 units from tech, .4 units from transportation and .2 units from mining.
- Each unit produced by mining requires .4 units from tech, .1 units from transportation and .2 units from mining.

Question:

1. What are intermediate demands created if tech plans to produce 100 units?
2. Determine the production levels needed to satisfy the following final demands:
   - 20 units from tech and no demand from the other sectors.
   - 20 units from tech, 30 from transportation and 40 from mining.

Solution: Since

- each unit produced by the tech sector requires .6 units from the tech sector, .1 from transportation and .2 from mining,
- each unit produced by transportation requires 0 units from tech, .4 units from transportation and .2 units from mining, and
- each unit produced by mining requires .4 units from tech, .1 units from transportation and .2 units from mining,

the consumption matrix is

$$C = \begin{bmatrix} .6 & 0 & .4 \\ .1 & .4 & .1 \\ .2 & .2 & .2 \end{bmatrix}.$$

Note that entries of C are nonnegative and column sums are < 1.

If tech plans to produce 100 units, the intermediate demands are given by

$$100 \begin{bmatrix} .6 \\ .1 \\ .2 \end{bmatrix} = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix} \begin{matrix} \leftarrow \text{60 units from technology} \\ \leftarrow \text{10 units from transportation} \\ \leftarrow \text{20 units from mining} \end{matrix}$$

Since we are asked to solve the problem for more than one final demand vector, we use the inverse matrix method.

We want to solve $(I - C)\vec{x} = \vec{d}$ and so we want to find $(I - C)^{-1}$.

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .6 & 0 & .4 \\ .1 & .4 & .1 \\ .2 & .2 & .2 \end{bmatrix} = \begin{bmatrix} .4 & 0 & -.4 \\ -.1 & .6 & -.1 \\ -.2 & -.2 & .8 \end{bmatrix}$$

We row reduce the superaugmented matrix.

$$\left[\begin{array}{c|c} I - C & I \end{array}\right] = \left[\begin{array}{ccc|ccc} .4 & 0 & -.4 & 1 & 0 & 0 \\ -.1 & .6 & -.1 & 0 & 1 & 0 \\ -.2 & -.2 & .8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{115}{32} & \frac{5}{8} & \frac{15}{8} \\ 0 & 1 & 0 & \frac{25}{32} & \frac{15}{8} & \frac{5}{8} \\ 0 & 0 & 1 & \frac{35}{32} & \frac{5}{8} & \frac{15}{8} \end{array}\right] \implies (I - C)^{-1} = \begin{bmatrix} \frac{115}{32} & \frac{5}{8} & \frac{15}{8} \\ \frac{25}{32} & \frac{15}{8} & \frac{5}{8} \\ \frac{35}{32} & \frac{5}{8} & \frac{15}{8} \end{bmatrix}$$

Remember the production equation:

$$(I - C)\vec{x} = \vec{d} \implies \vec{x} = (I - C)^{-1}\vec{d}$$

So if the final demand is 20 units from tech and no demand from the other sectors, the productions levels must be

$$\vec{x} = (I - C)^{-1} \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{115}{32} & \frac{5}{8} & \frac{15}{8} \\ \frac{25}{32} & \frac{15}{8} & \frac{5}{8} \\ \frac{35}{32} & \frac{5}{8} & \frac{15}{8} \end{bmatrix} \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 71.875 \\ 15.625 \\ 21.875 \end{bmatrix} \begin{matrix} \leftarrow \text{tech} \\ \leftarrow \text{transport} \\ \leftarrow \text{mining} \end{matrix}$$

If the final demand is 20 units from tech, 30 from transport and 40 from mining, the production levels must be

$$\vec{x} = (I - C)^{-1} \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} \frac{115}{32} & \frac{5}{8} & \frac{15}{8} \\ \frac{25}{32} & \frac{15}{8} & \frac{5}{8} \\ \frac{35}{32} & \frac{5}{8} & \frac{15}{8} \end{bmatrix} \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 165.625 \\ 96.875 \\ 115.625 \end{bmatrix} \begin{matrix} \leftarrow \text{tech} \\ \leftarrow \text{transport} \\ \leftarrow \text{mining} \end{matrix}$$

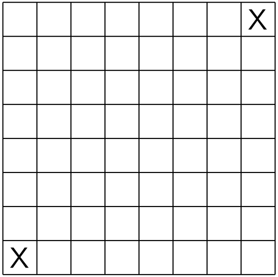
Warning!

Make sure you avoid this common mistake!

- When solving an Input-Output Model question by row reduction, you often scale the rows by factors of 10 to get rid of decimals. This is allowed because multiplying a row by a nonzero number is a valid row operation.

- However, when using the inverse matrix method, you cannot just multiply C or $I - C$ through by some powers of 10. This would change the coefficient matrix of your system (but not the constant terms), hence change its inverse, hence change your answer (to an incorrect one). You can only multiply rows through by powers of 10 in the superaugmented matrix (during row reduction).

Weekend problem



- Suppose you have an 8×8 checkerboard.
- Remove two diagonally opposite corners.
- Place dominoes on the board so that they cover two neighbouring (vertically or horizontally) squares. Dominoes cannot hang over the edge of the board.

Question: Can you cover every square without the dominoes overlapping?

Next time

For next time: Read Sections S, LISS, B.

- Subspaces of $\mathbb{R}^n$.
- Bases of subspaces.

Warning: The topic of subspaces is the one that students usually find the most difficult in the entire course. So it's especially important that you:

- Read Sections S, LISS, B before next lecture.
- Do the recommended exercises.
- Attend the DGDs for extra examples.