MAT 1302B – Mathematical Methods II

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Announcements

First Midterm:

- Solutions posted on course website.
- Handed back in DGDs this week.

Last time – matrix operations

- matrix addition, scalar multiplication
- matrix multiplication
  - sizes: $(m \times n)(n \times k) = (m \times k)$
  - Example:

$$\begin{bmatrix} 1 & 5 \\ -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 15 & 19 \\ 0 & 6 & 9 \\ 2 & 0 & -1 \end{bmatrix}$$

- matrix transpose
  - Example:

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 4 \end{bmatrix}$$

  - $(AB)^T = B^T A^T$
- matrix inverses

Last time – matrix inverses

Definition (Inverse of a matrix)

An $n \times n$ (square) matrix A is invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n.$$

We call C an inverse of A.

Terminology

- If a matrix A has an inverse, it has only one and we denote it A^{-1}.
- A matrix that is not invertible is sometimes called singular.
- An invertible matrix is sometimes called nonsingular.

Last time – inverses of 2×2 matrices

Theorem

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $ad - bc = 0$, then A is not invertible.

The determinant

The quantity $ad - bc$ is called the determinant of A and we write

$$\det A = ad - bc.$$

So a 2×2 matrix A is invertible iff $\det A \neq 0$.

Last time – using matrix inverses to solve a LS

Example: Solve the linear system

$$\begin{array}{rcrcr} 2x & - & 3y & = & -1 \\ -2x & + & 4y & = & 3 \end{array}$$

using the inverse matrix method.

Solution: We convert the problem to a matrix equation:

$$A\vec{x} = \vec{b}, \quad A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Then we find the inverse of A:

$$A^{-1} = \frac{1}{2 \cdot 4 - (-3)(-2)} \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 \\ 1 & 1 \end{bmatrix}$$

So

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & 3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 2 \end{bmatrix} \implies x = \frac{5}{2},\ y = 2.$$

Properties of inverses

Theorem

1. If A is invertible, then A^{-1} is invertible and
$$(A^{-1})^{-1} = A.$$
2. If A and B are $n \times n$ invertible matrices then AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}.$$
More generally, if $A_1, A_2, \ldots, A_k$ are all invertible then so is $A_1 A_2 \cdots A_k$ and
$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$
3. If A is invertible then A^T is invertible and
$$(A^T)^{-1} = (A^{-1})^T.$$

Justification of theorem

1. $(A^{-1})^{-1} = A$:
   $AA^{-1} = I = A^{-1}A \implies (A^{-1})^{-1} = A$

2. $(AB)^{-1} = B^{-1}A^{-1}$:

$$(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$$

   So $(AB)^{-1} = B^{-1}A^{-1}$.

3. $(A^T)^{-1} = (A^{-1})^T$:

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

   So $(A^T)^{-1} = (A^{-1})^T$.

Elementary matrices

Each elementary row operation can be expressed as left multiplication by an *elementary matrix*.

1. Replacement

$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\2a+d&2b+e&2c+f\\g&h&i\end{bmatrix}\qquad 2R_1+R_2$$

2. Interchange

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\g&h&i\\d&e&f\end{bmatrix}\qquad R_2\leftrightarrow R_3$$

3. Scaling

$$\begin{bmatrix}1&0&0\\0&-3&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\-3d&-3e&-3f\\g&h&i\end{bmatrix}\qquad -3R_2$$

The elementary matrices are invertible (remember that each row operation can be reversed).

Examples

1. Replacement

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Interchange

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = I$$

3. Scaling

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrices and inverses

Theorem

1. An $n \times n$ matrix A is invertible iff A is row equivalent to I_n.
2. If A is invertible, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Reason

Suppose A is invertible.

- If A is invertible, then $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$.
- Thus A has a pivot position in each row.
- Since A is square, it has a pivot position in each column.
- Thus its RREF is I_n.

Elementary matrices and inverses (cont.)

Reason (cont.)

Suppose A is row equivalent to I_n.

- Then there are elementary matrices $E_1, \dots, E_p$ such that

$$E_p \cdots E_2 E_1 A = I_n$$

$$\implies A^{-1} = E_p \cdots E_1 = E_p \cdots E_1 I_n$$

Finding matrix inverses

Algorithm for finding A^{-1}

- Form the superaugmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.
- Start row reducing to reduce A to RREF.
- If A is row equivalent to I, then $\left[\begin{array}{c|c} A & I \end{array}\right]$ is row equivalent to $\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$.
- Otherwise, A is not invertible.

Example 1

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

if it exists.

Solution: We row reduce the superaugmented matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_3+R_2]{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 4 & 0 & 3 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Therefore A is invertible and

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 3/4 & 1/4 & -3/4 \\ -1 & 0 & 1 \end{bmatrix}.$$

Check your answer!

$$\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 3/4 & 1/4 & -3/4 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

Note: Since A is square, we only need to check the multiplication in one direction.

Example 2

Find the inverse of

$$B = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 7 & 8 & -5 \\ 2 & -10 & 3 & 6 \\ 2 & 4 & -2 & 8 \end{bmatrix}$$

if it exists.

Solution: We row reduce the superaugmented matrix.

$$\left[\begin{array}{c|c} B & I \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 2 & -1 & 4 & 1 & 0 & 0 & 0 \\ 3 & 7 & 8 & -5 & 0 & 1 & 0 & 0 \\ 2 & -10 & 3 & 6 & 0 & 0 & 1 & 0 \\ 2 & 4 & -2 & 8 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_4} \left[\begin{array}{cccc|cccc} 1 & 2 & -1 & 4 & 1 & 0 & 0 & 0 \\ 3 & 7 & 8 & -5 & 0 & 1 & 0 & 0 \\ 2 & -10 & 3 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right]$$

B is not row equivalent to I and so B is not invertible.

Find matrix inverses - recap

- To find the inverse of A, row reduce the superaugmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.
- When row reducing, the operations we perform are determined by the left side of the superaugmented matrix (the right side "just comes along for the ride").
- If you get a row of zeros in the left matrix (left of the vertical bar), stop. The matrix A is not invertible.
- Otherwise, keep going until you get I on the left.
- Then the matrix on the right is A^{-1}.
- Check your answer by verifying that $A^{-1}A = I$ or $AA^{-1} = I$.

One more example of matrix inverses

Example: Find the inverse of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & -4 \\ -2 & -1 & 1 \end{bmatrix}.$$

Solution: We row reduce the superaugmented matrix.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & -4 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{4R_3+R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -1 & -4 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 4 & 1 & 4 \\ 0 & 1 & 0 & -3 & -1 & -4 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & -3 & -1 & -4 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

So A is invertible and

$$A^{-1} = \begin{bmatrix} 2 & \frac{1}{2} & 2 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}.$$

Check your answer!

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & -4 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2} & 2 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

Example 1

Suppose

$$A^2XBC = BEC$$

where all operations are defined and A, B, C are invertible. Solve for X.

Solution: Multiply on the right by C^{-1}:

$$A^2XBCC^{-1} = BECC^{-1} \implies A^2XBI = BEI \implies A^2XB = BE$$

Multiply on the right by B^{-1}:

$$A^2XBB^{-1} = BEB^{-1} \implies A^2XI = BEB^{-1} \implies A^2X = BEB^{-1}$$

Multiply on the left by $A^{-2} = (A^{-1})^2$:

$$A^{-2}A^2X = A^{-2}BEB^{-1} \implies IX = A^{-2}BEB^{-1} \implies X = A^{-2}BEB^{-1}$$

Example 2

Suppose we know

$$A^4 - 3A^3 + 2A^2 - 5I = 0.$$

Is A invertible?

Solution:

$$A^4 - 3A^3 + 2A^2 = 5I$$

$$A(A^3 - 3A^2 + 2A) = 5I$$

$$A\left(\frac{1}{5}(A^3 - 3A^2 + 2A)\right) = I$$

We could also factor from the other side to get

$$\frac{1}{5}(A^3 - 3A^2 + 2A)A = I.$$

Thus A is invertible and

$$A^{-1} = \frac{1}{5}(A^3 - 3A^2 + 2A).$$

Example 3

Expand $(A+B)^2$.

Solution:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B) \\ &= A(A+B) + B(A+B) \\ &= A^2 + \underbrace{AB + BA}_{\text{can't combine!}} + B^2\end{aligned}$$

Example 4

Simplify

$$C^{-1}(AB)^T(A^{-1})^T(B^{-1})^T C^2.$$

Solution:

$$\begin{aligned}
C^{-1}(AB)^T(A^{-1})^T(B^{-1})^T C^2 &= C^{-1}B^T A^T (A^T)^{-1}(B^T)^{-1}C^2 \\
&= C^{-1}B^T I (B^T)^{-1} C^2 \\
&= C^{-1}B^T (B^T)^{-1} C^2 \\
&= C^{-1} I C^2 \\
&= C^{-1} C^2 \\
&= C^{-1} C C \\
&= I C \\
&= C
\end{aligned}$$

Example 5

For which values of a is the following matrix invertible?

$$\begin{bmatrix} 3 & 5 & a \\ 0 & 2 & -3 \\ -3 & -1 & a \end{bmatrix}$$

Solution: We know that the matrix is invertible iff it is row equivalent to I. So we row reduce.

$$\begin{bmatrix} 3 & 5 & a \\ 0 & 2 & -3 \\ -3 & -1 & a \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 3 & 5 & a \\ 0 & 2 & -3 \\ 0 & 4 & 2a \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 3 & 5 & a \\ 0 & 2 & -3 \\ 0 & 0 & 2a+6 \end{bmatrix}$$

A square matrix is row equivalent to I iff it has a pivot in each row/column.

So the matrix is invertible exactly when $2a + 6 \neq 0$. That is, when $a \neq -3$.

Weekend problem – from last time

1. Pick a number between 1 and 9.
2. Subtract 5.
3. Take the absolute value (i.e. if your number is negative, forget the negative sign).
4. Add 1.
5. Multiply by 3.
6. Add 3.
7. Multiply by 3.
8. If your number is more than one digit, add the digits together (repeat this process until you have a single digit).
9. Subtract 5.
10. Convert your number to a letter in the alphabet ($A = 1$, $B = 2$, etc.)
11. Think of a country that begins with that letter.
12. Take the second letter of your country.
13. Think of an animal that begins with that letter.
14. Think of the colour of your animal.

Weekend problem – Solution

How does it work?

- Steps 1–3 are just to make the numbers easier to work with.
- Steps 4–7 are to ensure you have a multiple of 9.
- Step 8: the sum of the digits of a multiple of 9 is again a multiple of 9. So after step 8, everyone has the number 9.
- After Step 9, everyone has 4.
- The country beginning with D that most people pick is Denmark.
- The animal beginning with E that most people pick is elephant.

Next time

Next time:

- Leontief Input-Output Model.
- Application of matrix equations to economics.