MAT 1302B – Mathematical Methods II

Alistair Savage

Mathematics and Statistics
University of Ottawa

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Announcements

First Midterm:

- Solutions posted on course website.
- Handed back in DGDs this week.

Last time – matrix operations

- matrix addition, scalar multiplication
- matrix multiplication
  - sizes: $(m \times n)(n \times k) =$
  - Example:

$$\begin{bmatrix} 1 & 5 \\ -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

- matrix transpose
  - Example:

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 4 \end{bmatrix}^T =$$

  - $(AB)^T =$
- matrix inverses

Last time – matrix inverses

Definition (Inverse of a matrix)

An $n \times n$ (square) matrix A is invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n.$$

We call C an inverse of A.

Terminology

- If a matrix A has an inverse, it has only one and we denote it A^{-1}.
- A matrix that is not invertible is sometimes called singular.
- An invertible matrix is sometimes called nonsingular.

Last time – inverses of 2×2 matrices

Theorem

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $ad - bc = 0$, then A is not invertible.

The determinant

The quantity $ad - bc$ is called the determinant of A and we write

$$\det A = ad - bc.$$

So a 2×2 matrix A is invertible iff $\det A \neq 0$.

Last time – using matrix inverses to solve a LS

Example: Solve the linear system

$$\begin{array}{rcrcr} 2x & - & 3y & = & -1 \\ -2x & + & 4y & = & 3 \end{array}$$

using the inverse matrix method.

Solution: We convert the problem to a matrix equation:

$$A\vec{x} = \vec{b}, \quad A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Then

So

$$\vec{x} =$$

Properties of inverses

Theorem

1. If A is invertible, then

2. If A and B are $n \times n$ invertible matrices then

3. If A is invertible then

Justification of theorem

Elementary matrices

Each elementary row operation can be expressed as left multiplication by an *elementary matrix*.

1. Replacement

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

2. Interchange

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

3. Scaling

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

The elementary matrices are invertible (remember that each row operation can be reversed).

Examples

1. Replacement

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

2. Interchange

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} = I$$

3. Scaling

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{-1}{3} & 0\\ 0 & 0 & 1\end{bmatrix} = I = \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{-1}{3} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Elementary matrices and inverses

Theorem

1. An $n \times n$ matrix A is invertible iff A is row equivalent to I_n.
2. If A is invertible, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Reason

Suppose A is invertible.

- If A is invertible, then $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$.
- Thus A has a pivot position in each row.
- Since A is square, it has a pivot position in each column.
- Thus its RREF is I_n.

Elementary matrices and inverses (cont.)

Reason (cont.)

Suppose A is row equivalent to I_n.

- Then there are elementary matrices $E_1, \dots, E_p$ such that

$$E_p \cdots E_2 E_1 A = I_n$$

$$\implies A^{-1} = E_p \cdots E_1 = E_p \cdots E_1 I_n$$

Finding matrix inverses

Algorithm for finding A^{-1}

- Form the superaugmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.
- Start row reducing to reduce A to RREF.
- If A is row equivalent to I, then $\left[\begin{array}{c|c} A & I \end{array}\right]$ is row equivalent to $\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$.
- Otherwise, A is not invertible.

Example 1

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

if it exists.

Solution:

Therefore

Example 2

Find the inverse of

$$B = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 7 & 8 & -5 \\ 2 & -10 & 3 & 6 \\ 2 & 4 & -2 & 8 \end{bmatrix}$$

if it exists.

Solution:

Find matrix inverses - recap

- To find the inverse of A, row reduce the superaugmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.

- When row reducing, the operations we perform are determined by the left side of the superaugmented matrix (the right side "just comes along for the ride").

- If you get a row of zeros in the left matrix (left of the vertical bar), stop. The matrix A is not invertible.

- Otherwise, keep going until you get I on the left.

- Then the matrix on the right is A^{-1}.

- Check your answer by verifying that $A^{-1}A = I$ or $AA^{-1} = I$.

One more example of matrix inverses

Example: Find the inverse of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & -4 \\ -2 & -1 & 1 \end{bmatrix}.$$

Solution:

So

Example 1

Suppose

$$A^2XBC = BEC$$

where all operations are defined and A, B, C are invertible. Solve for X.

Solution:

Example 2

Suppose we know

$$A^4 - 3A^3 + 2A^2 - 5I = 0.$$

Is A invertible?

Solution:

Example 3

Expand $(A + B)^2$.

Solution:

Example 4

Simplify

$$C^{-1}(AB)^T(A^{-1})^T(B^{-1})^TC^2.$$

Solution:

Example 5

For which values of a is the following matrix invertible?

$$\begin{bmatrix} 3 & 5 & a \\ 0 & 2 & -3 \\ -3 & -1 & a \end{bmatrix}$$

Solution:

Weekend problem – from last time

1. Pick a number between 1 and 9.
2. Subtract 5.
3. Take the absolute value (i.e. if your number is negative, forget the negative sign).
4. Add 1.
5. Multiply by 3.
6. Add 3.
7. Multiply by 3.
8. If your number is more than one digit, add the digits together (repeat this process until you have a single digit).
9. Subtract 5.
10. Convert your number to a letter in the alphabet ($A = 1$, $B = 2$, etc.)
11. Think of a country that begins with that letter.
12. Take the second letter of your country.
13. Think of an animal that begins with that letter.
14. Think of the colour of your animal.

Weekend problem – Solution

How does it work?

Next time

Next time:

- Leontief Input-Output Model.
- Application of matrix equations to economics.