MAT 1302B – Mathematical Methods II

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Announcements

Midterm 1:

- Solutions posted on website.
- Will be handed back in DGDs after reading break.
- Marks will be posted on Blackboard Learn.

Last time – matrix arithmetic

- Zero matrix: a matrix with all entries equal to zero.
- Identity matrix: a square matrix with 1s on the main diagonal and zeros everywhere else:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

- Matrix addition: $A + B$
  - Can only add matrices of the same size.
  - Add matrices by adding corresponding entries.
- Scalar multiplication: rA, $r \in \mathbb{R}$, A a matrix.
  - Multiply each entry by r.
- Linear combinations: $2A + \frac{1}{2}B + \frac{2}{5}C$

Example

Suppose

$$A = \begin{bmatrix} 6 & 4 & -2 \\ 8 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned} \frac{1}{2}A + B - 2C &= \frac{1}{2}\begin{bmatrix} 6 & 4 & -2 \\ 8 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 2 \\ 2 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 & -2 \\ 4 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Last time – matrix multiplication

Sizes:

$$(m \times n)(n \times k) = (m \times k)$$

Original definition:

$$AB = A \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_k \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_k \end{bmatrix}$$

Shorter computation: To compute the (i, j) entry (entry in i-th row and j-th column) in the product AB:

1. Look at i-th row in lefthand matrix A and j-th column in righthand matrix B (must have same number of entries).
2. Move right along the row and down the column.
3. Multiply corresponding entries and add the products.

Matrix multiplication - demonstration

http://demonstrations.wolfram.com/MatrixMultiplication/

Matrix multiplication example 1:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 2 + (-1)3 & 1 \cdot 1 + 0(-2) + (-1)1 \\ 2 \cdot 0 + 1 \cdot 2 + 0 \cdot 3 & 2 \cdot 1 + 1(-2) + 0 \cdot 1 \\ (-2)0 + 3 \cdot 2 + 1 \cdot 3 & (-2)1 + 3(-2) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 2 & 0 \\ 9 & -7 \end{bmatrix}$$

Matrix multiplication example 2:

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & -1 \\ 1 & 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & -5 \\ 7 & 5 & 5 & 2 \\ 5 & 5 & 5 & 4 \end{bmatrix}$$

Matrix multiplication example 3:

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ 2 & 11 \end{bmatrix}$$

Properties of matrix multiplication

Theorem

Suppose A is an $m \times n$ matrix and B and C have sizes so that the following operations are defined. Then

1. $A(BC) = (AB)C$,
2. $A(B + C) = AB + AC$,
3. $(B + C)A = BA + CA$,
4. $r(AB) = (rA)B = A(rB)$ for any scalar r,
5. $I_m A = A = A I_n$.

Properties of matrix multiplication

Justification: $I_m A = A = A I_n$

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$. Then $I_m A = \begin{bmatrix} I_m \vec{a}_1 & \cdots & I_m \vec{a}_n \end{bmatrix}$.

$$I_m \vec{a}_i = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} = a_{1i} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_{2i} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_{mi} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} = \vec{a}_i$$

So

$$I_m A = \begin{bmatrix} I_m \vec{a}_1 & \cdots I_m \vec{a}_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} = A.$$

Showing that $A I_n = A$ is similar (exercise).

Properties of matrix multiplication

Justification: $A(B + C) = AB + AC$

Let $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_k \end{bmatrix}$ and $C = \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_k \end{bmatrix}$. Then

$$\begin{aligned} A(B + C) &= A \begin{bmatrix} (\vec{b}_1 + \vec{c}_1) & \cdots & (\vec{b}_k + \vec{c}_k) \end{bmatrix} \\ &= \begin{bmatrix} A(\vec{b}_1 + \vec{c}_1) & \cdots & A(\vec{b}_k + \vec{c}_k) \end{bmatrix} \\ &= \begin{bmatrix} (A\vec{b}_1 + A\vec{c}_1) & \cdots & (A\vec{b}_k + A\vec{c}_k) \end{bmatrix} \\ &= \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_k \end{bmatrix} + \begin{bmatrix} A\vec{c}_1 & \cdots & A\vec{c}_k \end{bmatrix} \\ &= AB + AC \end{aligned}$$

Justification of other properties can be found in the text.

Warnings!!

1. In general $AB \neq BA$ (even if both defined).

   Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Definition

If $AB = BA$ (for a **specific** A and B), we say A and B commute.

Example

I_n commutes with all $n \times n$ matrices, since

$$AI_n = A = I_n A.$$

Warnings!! (cont.)

2. Cancelation law does not hold for matrices.

   I.e. $AB = AC$ does not imply $B = C$ (even if A is not the zero matrix).

   Example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 0 & 0 \end{bmatrix}, \quad \text{but} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 5 & 7 \\ 0 & 0 \end{bmatrix}.$$

3. $AB = 0$ does not imply $A = 0$ or $B = 0$.

   Example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Powers of a matrix

If k is a positive integer, we define

$$A^k = \overbrace{AA \cdots A}^{k \text{ times}}.$$

We also define

$$A^0 = I \quad \text{(the identity matrix)}.$$

Note: Powers of a matrix are only defined if the matrix is square since only square matrices can be multiplied by themselves.

Transpose

Definition

If A is an $m \times n$ matrix, then its transpose A^T is the $n \times m$ matrix whose i-th column is the i-th row of A (and whose i-th row is the i-th column of A). Sometimes the transpose is denoted A^t.

Alternatively, the (i, j) entry of A^T is the (j, i) entry of A.

Examples

1. $$A = \begin{bmatrix} 2 & 3 & -1 \\ 8 & 0 & 7 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ -1 & 7 \end{bmatrix}$$

2. $$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Properties of the transpose

Theorem

Suppose A and B are matrices such that the following operations are defined. Then

1. $(A^T)^T = A$,
2. $(A+B)^T = A^T + B^T$,
3. $(rA)^T = rA^T$ for any scalar r,
4. $(AB)^T = B^T A^T$ (note that the order changes).

   More generally,

$$(A_1 A_2 \cdots A_k)^T = A_k^T A_{k-1}^T \cdots A_1^T.$$

Note: $(AB)^T$ is usually not equal to $A^T B^T$ (it is equal only when A and B commute).

Inverse of a matrix

Real numbers

Nonzero real numbers have multiplicative inverses: if $c \neq 0$,

$$cc^{-1} = c^{-1}c = 1.$$

E.g.

$$3 \cdot 3^{-1} = 3^{-1} \cdot 3 = 1.$$

We want to consider an analogue for matrices.

Definition (Inverse of a matrix)

An $n \times n$ (square) matrix A is invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n.$$

We call C an inverse of A.

Uniqueness of inverses

Suppose B and C are both inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

So inverses are unique (i.e. a matrix has at most one inverse)!

If it exists, we denote the (unique) inverse of A by A^{-1}. So

$$A^{-1}A = I = AA^{-1}.$$

Note

For non-square matrices, can have $AB = I$ and $BA \neq I$.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I.$$

But this can't happen for square matrices (so it's enough to check the inverse property in one order).

Example

$$\begin{bmatrix} 3 & 8 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 & 8 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} -5 & 8 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus,

$$\begin{bmatrix} 3 & 8 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 8 \\ 2 & -3 \end{bmatrix}.$$

Terminology

- A matrix that is not invertible is sometimes called singular.
- An invertible matrix is sometimes called nonsingular.
- If k is a positive integer and A is an invertible matrix, we define

$$A^{-k} = \overbrace{A^{-1}A^{-1}\cdots A^{-1}}^{k \text{ times}}.$$

Inverses of 2×2 matrices

Theorem

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $ad - bc = 0$, then A is not invertible.

The determinant

The quantity $ad - bc$ is called the determinant of A and we write

$$\det A = ad - bc.$$

So a 2×2 matrix A is invertible iff $\det A \neq 0$.

Inverses of 2×2 matrices (cont.)

Justification

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & -3 \\ -5 & 9 \end{bmatrix}$$

$$\det A = 2 \cdot 9 - (-3)(-5) = 18 - 15 = 3 \neq 0$$

Thus A is invertible and

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 9 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ \frac{5}{3} & \frac{2}{3} \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$$

$$\det A = 2(-3) - (-6)1 = 0$$

Thus A is not invertible (it has no inverse).

Uses of multiplicative inverses

We can use multiplicative inverses to solve simple equations.

Example

Suppose we want to solve the equation (in real numbers) $ax = b$ for x. If $a \neq 0$, we multiply both sides by a^{-1}:

$$a^{-1}ax = a^{-1}b \implies x = a^{-1}b.$$

Can we do the same thing with matrix equations?

Example

Suppose we want to solve the following equation for $\vec{x}$:

$$A\vec{x} = \vec{b}$$

If A is invertible, we multiply on both sides on the left by A^{-1}:

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \implies I\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}$$

Uses of multiplicative inverses

Important notes

1. For the case of real numbers, we only need the coefficient to be nonzero (then we can multiply by the inverse).
2. For matrix equations, the matrix must be invertible (nonzero is not enough!).
3. For matrix equations, we must multiply both sides of the equation on a particular side.

Application to linear systems

Suppose we want to solve the linear system

$$\begin{array}{rcrcr} 2x & - & 3y & = & 3 \\ -5x & + & 9y & = & -6 \end{array}$$

This system is equivalent to

$$A\vec{x} = \vec{b}, \quad \text{where} \quad A = \begin{bmatrix} 2 & -3 \\ -5 & 9 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}.$$

Since A is invertible (from earlier), we can multiply on the left by A^{-1}.

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \implies I\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}$$

So

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2 \cdot 9 - (-3)(-5)} \begin{bmatrix} 9 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ \frac{5}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Thus the unique solution is $(x, y) = (3, 1)$.

Using invertible matrices to solve linear systems

Theorem

If A is an $n \times n$ invertible matrix, then for each $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

Notes

1. This technique only works if the coefficient matrix is invertible (whereas row reduction always works).
2. Note that invertible matrices are square. Thus this technique has no hope of working if the coefficient matrix is not square (i.e. must have the same number of equations as variables).
3. In practice, row reduction is often a faster procedure for solving a LS than finding the inverse matrix.
4. However, once we know A^{-1}, it is easy to solve LS's with the same coefficient matrix A but different $\vec{b}$'s (don't have to do row reduction over and over).

Weekend problem

1. Pick a number between 1 and 9.
2. Subtract 5.
3. Take the absolute value (i.e. if your number is negative, forget the negative sign).
4. Add 1.
5. Multiply by 3.
6. Add 3.
7. Multiply by 3.
8. If your number is more than one digit, add the digits together (repeat this process until you have a single digit).
9. Subtract 5.
10. Convert your number to a letter in the alphabet ($A = 1$, $B = 2$, etc.)
11. Think of a country that begins with that letter.
12. Take the second letter of your country.
13. Think of an animal that begins with that letter.
14. Think of the colour of your animal.

Weekend problem

You have a grey elephant from Denmark.

Question: How does this work?

Next time

For next time: Read Sections MISLE, MINM.

- More about matrix inverses.
- How to find the inverse of an invertible matrix of any (square) size.