MAT 1302B – Mathematical Methods II

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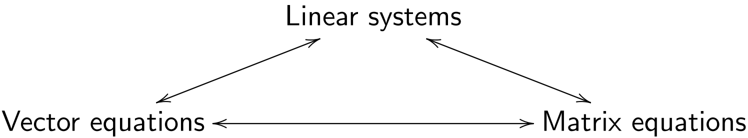
Recap

Three languages:



We know

1. How to translate between different languages.
2. How to find solutions sets in all the languages (key step is always row reduction).
3. How to express solutions sets in all the languages.

Recap: Linear dependence/independence

Definition

Consider a list of vectors $\vec{v}_1, \dots, \vec{v}_p$ in $\mathbb{R}^n$ and the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}. \tag{1}$$

- If (1) has only the trivial solution, the list of vectors is linearly independent.
- If (1) has a nontrivial solution, the list of vectors is linearly dependent.

Put another way, the list of vectors is linearly dependent if there exist some scalars $c_1, \dots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}. \tag{2}$$

(2) is called a linear dependence relation.

Recap: Linear dependence/independence

The most important thing to remember about linear dependence and independence is:

If you're given a list of vectors

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$$

and you want to know if they're linearly dependent or independent, you:

1. Form the matrix
$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix}$$
with these vectors as its columns.
2. Row reduce $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$.
3. Then
   - if there are free variables, the vectors are dependent, and
   - if there are no free variables, the vectors are independent.
4. If they are linearly dependent and you want a linear dependence relation, find any nontrivial solution by picking any values for the free variables (not all zero).

Recap example: linear independence

Question: Are the following vectors linearly dependent or independent? If they are dependent, find a linear dependence relation.

$$(1,2,4), \quad (-1,3,1), \quad (2,-1,3)$$

Solution: We row reduce

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 0 \\ 4 & 1 & 3 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is one free variable, the vectors are linearly dependent. The general solution to the corresponding system is:

$$x_1 = -x_3, \quad x_2 = x_3, \quad x_3 \text{ free.}$$

To find a linear dependence relation, choose any nonzero value for the free variable, say $x_3 = 1$. Then $x_1 = -1$ and $x_2 = 1$. So a linear dependence relation is

$$-(1,2,4) + (-1,3,1) + (2,-1,3) = (0,0,0). \qquad \text{Check!}$$

Overview

Goal for next few lectures

Further develop the language of matrices – “matrix arithmetic”.

Notation/terminology

Suppose A is an $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & & & \vdots & & \vdots \\ \vdots & & & \vdots & & \vdots \\ a_{i1} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & \vdots & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$$

Here

- $\vec{a}_j$ denotes the j-th column
- We number rows/columns from top/left

$$\begin{aligned} a_{ij} &= \text{entry in } i\text{-th row and } j\text{-th column} \\ &= i\text{-th entry (from top) in } \vec{a}_j \end{aligned}$$

We sometimes write $A = [a_{ij}]$ or A_{ij} for a_{ij}.

Example

If

$$A = [a_{ij}] = \begin{bmatrix} 2 & 7 & 8 & -5 & 10 \\ 3 & 6 & -4 & 9 & 5 \\ -3 & 0 & 0 & 1 & 8 \end{bmatrix}$$

then

- $a_{12} = 7$,
- $a_{34} = 1$,
- $a_{25} = 5$,
- a_{41} – no such entry.

Notation/terminology (cont.)

- Diagonal entries of A are the entries $a_{11}, a_{22}, \ldots$ and the form the main diagonal.
- Diagonal matrix: square (i.e. $n \times n$ for some n) matrix such that nondiagonal entries are zero.
  E.g.
  $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- $n \times n$ identity matrix:
  $$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Notation/terminology (cont.)

- Zero matrix: a matrix whose entries are all zero.
  - Denote the $m \times n$ zero matrix by $0_{m\times n}$ or $0_{m,n}$.
  - When size is clear from context, we sometimes just write 0.
- Equality: two matrices are equal if they are the same size and their corresponding entries are equal.

Examples:

- $$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$
- $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix addition

Definition

If A and B are both $m \times n$ matrices, then $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B. In other words,

$$A + B = C \qquad \text{where} \qquad c_{ij} = a_{ij} + b_{ij}.$$

Example

Suppose $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 8 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 7 \\ -8 & 10 & \frac{11}{3} \end{bmatrix}$, $C = \begin{bmatrix} -2 & 3 \\ 5 & -2 \\ -5 & 3 \end{bmatrix}$.

Then

- $A + B$ and $B + C$ are not defined.
- $A + C = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 5 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 4 & 1 \\ 3 & 5 \end{bmatrix}$

Scalar multiplication

Definition

If r is a scalar and A is a matrix, then rA is the matrix obtained from A by multiplying all entries of A by r.

Example

$$2\begin{bmatrix} 3 & -1 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 10 & 14 \end{bmatrix}, \quad 0\begin{bmatrix} 1 & 2 \\ -10 & 12 \\ 15 & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{3\times 2}$$

Definition

We define

- $-A \stackrel{\text{def}}{=} (-1)A$ (negative of a matrix)
- $A - B \stackrel{\text{def}}{=} A + (-1)B$ (subtraction of matrices)

Example

Suppose

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 8 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 7 \\ -8 & 10 & \frac{11}{3} \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 \\ 5 & -2 \\ -5 & 3 \end{bmatrix}.$$

Then

- $2A - C = \begin{bmatrix} 4 & 0 \\ -2 & 6 \\ 16 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 3 \\ 5 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -7 & 8 \\ 21 & 1 \end{bmatrix}$
- $3A + 2B$ not defined

Note

- Remember that vectors are just matrices with one column.
- We defined vector addition and scalar multiplication for vectors.
- Those vector operations are just special cases of matrix operations.

Properties of matrix operations

Recall: Vector operations satisfied a list of properties.

Matrix operations satisfy similar properties.

Theorem

Suppose A, B and C are matrices of the same size and r and s are scalars. Then

1. $A + B = B + A$ — commutativity
2. $(A + B) + C = A + (B + C)$ — associativity
3. $A + 0 = 0 + A = A$ — 0 is an additive identity
4. $r(A + B) = rA + rB$ — distributivity I
5. $(r + s)A = rA + sA$ — distributivity II
6. $r(sA) = (rs)A$ — associativity

Matrix multiplication

Question

How should we define the product of two matrices?

Key is to think of a matrix as a map:

$$\vec{v} \xrightarrow{\text{mult by } A} A\vec{v}$$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix.

Matrices as maps

Question: What kind of vectors can we multiply B by?

Answer: In order for $B\vec{v}$ to be defined, we must have $\vec{v} \in \mathbb{R}^p$.

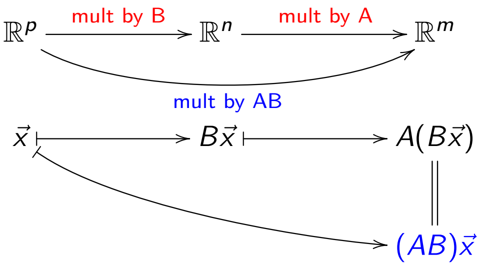
Question: If $\vec{v} \in \mathbb{R}^p$, what type of vector is $B\vec{v}$?

Answer: Remember, the number of entries in $B\vec{v}$ is equal to the number of rows of B. So $B\vec{v} \in \mathbb{R}^n$.

So multiplication by the $n \times p$ matrix B takes vectors in $\mathbb{R}^p$ to vectors in $\mathbb{R}^n$.

Similarly, multiplication by the $m \times n$ matrix A takes vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$

So we have the maps



We want to define the product AB of matrices so that multiplying a vector by AB has the same effect as multiplying it by B and then A.

Matrix multiplication

- Suppose $\vec{x} \in \mathbb{R}^p$ and $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix}$ is an $n \times p$ matrix.
- Then $B\vec{x} = x_1\vec{b}_1 + \cdots + x_p\vec{b}_p$.
- So

$$\begin{aligned} A(B\vec{x}) &= A(x_1\vec{b}_1 + \cdots + x_p\vec{b}_p) \\ &= A(x_1\vec{b}_1) + \cdots + A(x_p\vec{b}_p) \\ &= x_1A\vec{b}_1 + \cdots + x_pA\vec{b}_p \\ &= \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix} \vec{x}. \end{aligned}$$

- So we define

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}.$$

Matrix multiplication

Definition

Suppose A is an $m \times n$ matrix and $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix}$ is an $n \times p$ matrix.

Then AB is the $m \times p$ matrix with columns $A\vec{b}_1, \dots, A\vec{b}_p$.

$$AB = A \begin{bmatrix} \vec{b}_1 & \dots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \dots & A\vec{b}_p \end{bmatrix}$$

So to compute the product AB, we multiply every column of B by A.

Example

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 \\ -1 & -2 & 0 \\ 2 & 1 & 4 \end{bmatrix}$$

We have

$$A\vec{b}_1 = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 0 \cdot (-1) + (-1) \cdot 2 \\ 3 \cdot 1 + 1 \cdot (-1) + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix},$$

$$A\vec{b}_2 = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad A\vec{b}_3 = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 17 \end{bmatrix}.$$

So

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & 17 \end{bmatrix}.$$

Matrix sizes and multiplication

Important notes

- In order for the product AB to be defined, we must have

$$\#\text{ columns of } A = \#\text{ rows of } B$$

- Note what determines the size of the product:

$$(m \times n \text{ matrix})(n \times p \text{ matrix}) = m \times p \text{ matrix}$$

Matrix-vector product

The above matches what we learned about the matrix-vector product.

An $m \times n$ matrix can be multiplied by vectors in $\mathbb{R}^n$, and the result is a vector in $\mathbb{R}^m$.

$$(m \times n \text{ matrix})(n \times 1 \text{ vector}) = m \times 1 \text{ vector}$$

Examples

In the following, are AB and BA defined and if so, what are their sizes/dimensions?

1. A is 7×5, B is 5×2.

   AB is defined and is 7×2. BA is not defined.

2. A is 4×6, B is 8×4.

   AB is not defined. But BA is defined and is 8×6.

3. A is 4×5 and B is 6×7.

   Neither AB nor BA is defined.

4. A is 2×9 and B is 9×2.

   AB is defined and is 2×2. BA is also defined and is 9×9.

5. A and B are both 5×5.

   AB and BA are both defined and are both 5×5.

Direction computation of matrix products

Direct definition

If AB is defined, then the entry of AB in row i and column j is the sum of the products of corresponding entries in row i of A and column j of B:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 2 & 0 \end{bmatrix}, \quad C = AB \quad 2 \times 2$$

To compute c_{12}, look at row 1 of A and column 2 of B:

$$c_{12} = (-1)(-3) + 2 \cdot 5 + 0 \cdot 0 = 13$$

Doing this for the other entries, we get:

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (-1)1 + 2 \cdot 0 + 0 \cdot 2 & 13 \\ 1 \cdot 1 + 4 \cdot 0 + (-3)2 & 1(-3) + 4 \cdot 5 + (-3)0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 13 \\ -5 & 17 \end{bmatrix}$$

Examples

If

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 2 \cdot 0 + 0 \cdot 2 & 2 \cdot 3 + 0 \cdot 1 \\ (-1) \cdot 0 + 1 \cdot 2 & (-1) \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 2 & -2 \end{bmatrix}.$$

If

$$M = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -1 & -1 \end{bmatrix},$$

then

$$MN = \begin{bmatrix} 3 \cdot 2 + 0 \cdot 1 + 1 \cdot (-1) & 3 \cdot 0 + 0 \cdot 1 + 1 \cdot (-1) \\ 0 \cdot 2 + 5 \cdot 1 + (-1) \cdot (-1) & 0 \cdot 0 + 5 \cdot 1 + (-1) \cdot (-1) \\ (-1) \cdot 2 + 1 \cdot 1 + 0 \cdot (-1) & (-1) \cdot 0 + 1 \cdot 1 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 6 & 6 \\ -1 & 1 \end{bmatrix}$$

Warning!!!!

In general $AB \neq BA$!! Three things can happen.

1. It's possible that one product is defined and the other is not.

   Example: If A is 3×2 and B is 2×4, then AB is defined, but BA is not (sizes don't match).

2. Even if both products are defined, they may be different sizes.

   Example: If A is 3×2 and B is 2×3, then AB is 3×3 but BA is 2×2.

3. If both are square of the same size, both products are defined and have the same size but still may not be equal.

   Example: If

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   then

   $$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Next time

Read: Sections MISLE, MINM

Next time, we will continue with matrix arithmetic.