MAT 1302B – Mathematical Methods II

Alistair Savage

Mathematics and Statistics
University of Ottawa

Winter 2015 – Lecture 7

Last time

- General solutions (with free variables) to matrix (or vector) equations.
- Wrote solution sets in vector parametric form.
- Saw that homogeneous systems play an important role in such descriptions.
- Discussed relation between the solution set of a LS and the corresponding homogeneous system.

Last time: Homogeneous linear systems

Definition

A LS is homogeneous if it can be written in the form

$$A\vec{x} = \vec{0}$$

where A is an $m \times n$ matrix and $\vec{0} \in \mathbb{R}^m$.

A homogenous system always has the solution $\vec{x} = \vec{0} \in \mathbb{R}^n$, called the trivial solution.

A nontrivial solution is a solution $\vec{x} \neq \vec{0}$.

Theorem

The homogeneous equation $A\vec{x} = \vec{0}$ has a nontrivial solution iff it has at least one free variable.

Last time: Solutions to homogeneous systems

Description in terms of Span

The solution set of a homogeneous system can always be written in the form

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$$

for some vectors $\vec{v}_1, \ldots, \vec{v}_p$.

Theorem

If $A\vec{x} = \vec{b}$ is consistent and $\vec{p}$ is any solution, then full the solution set is the set of all vectors of the form

$$\vec{w} = \vec{p} + \vec{v}_h$$

where $\vec{v}_h$ is any solution of the corresponding homogeneous equation $A\vec{x} = \vec{0}$. (Note: $\vec{p}$ is called a particular solution.)

Example

Suppose

$$A = \begin{bmatrix} 1 & -3 & -4 & -4 \\ -2 & 6 & 6 & 4 \\ -1 & 3 & 2 & 0 \\ 3 & -9 & -10 & -8 \end{bmatrix}.$$

Find the general solution to the homogeneous equation $A\vec{x} = \vec{0}$.

Solution: We row reduce.

$$\left[\begin{array}{cccc|c} 1 & -3 & -4 & -4 & 0 \\ -2 & 6 & 6 & 4 & 0 \\ -1 & 3 & 2 & 0 & 0 \\ 3 & -9 & -10 & -8 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is then:

$$x_1 = 3x_2 - 4x_4$$
$$x_3 = -2x_4$$
$$x_2, x_4 \text{ free}$$

Example (cont.)

$$\begin{aligned} x_1 &= 3x_2 - 4x_4 \\ x_3 &= -2x_4 \\ x_2, x_4 &\text{ free} \end{aligned}$$

We want our solution in vector parametric form.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 4x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}$$

So the solution set is:

$$\text{Span}\{(3, 1, 0, 0), (-4, 0, -2, 1)\}$$

Example (cont.)

$$A = \begin{bmatrix} 1 & -3 & -4 & -4 \\ -2 & 6 & 6 & 4 \\ -1 & 3 & 2 & 0 \\ 3 & -9 & -10 & -8 \end{bmatrix}$$

Solution set to $A\vec{x} = \vec{0}$: $\text{Span}\{(3,1,0,0),(-4,0,-2,1)\}$

New Question: Let

$$\vec{p} = (0,1,0,0), \quad \vec{b} = A\vec{p} = (-3,6,3,-9) \quad \text{(2nd column of } A\text{)}.$$

What is the general solution to the nonhomogeneous equation $A\vec{x} = \vec{b}$?

Solution: We just add the particular solution $\vec{p}$ to the solutions for the homogeneous equation $A\vec{x} = \vec{0}$!

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}$$

Question: Does the following vector equation have a solution?

$$x_1 \begin{bmatrix} 2 \\ -2 \\ -4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \\ 6 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 4 \\ -7 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ -3 \\ 17 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer: YES! It is homogeneous and thus has the trivial solution $\vec{x} = \vec{0}$.

Question: Does it have nontrivial solutions?

Solution: We row reduce to echelon form:

$$\left[\begin{array}{ccccc|c} 2 & -2 & 1 & -1 & -5 & 0 \\ 2 & -1 & 4 & 0 & -3 & 0 \\ -4 & 6 & 4 & 5 & 17 & 0 \\ -2 & 0 & -7 & 0 & 4 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccccc|c} 2 & -2 & 1 & -1 & -5 & 0 \\ 0 & 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are free variables, the answer is YES, there are nontrivial solutions.

Today: Overview

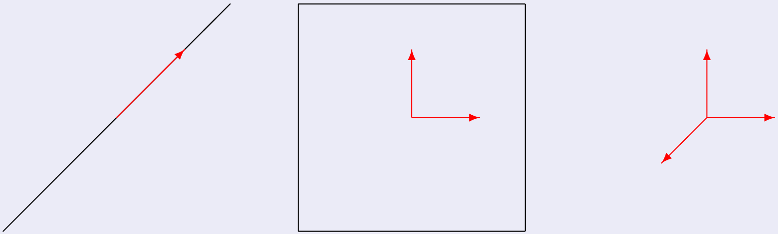
- Look at homogeneous systems in the language of vector equations.
- Examine an important notion: linear dependence of vectors.
- Later, we'll see that linear dependence is closely related to dimension.

Motivation: dimension

What is dimension?

Intuitive idea: Dimension is how many directions we need to "get everywhere" in a space.

Line | Plane | Space

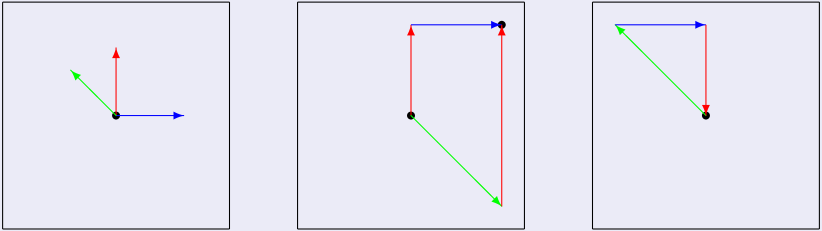


One dimensional | Two dimensional | Three dimensional

Motivation: dimension

Problem

What about having three directions in a plane?



Resolution

- One direction is not needed (is redundant).
- With three vectors, there are multiple ways to get to the same place (even if we use a multiple of each vector only once).
- Alternatively: We can return to the origin, using (a multiple of) each vector only once.

Linear dependence

Definition

Consider a list of vectors $\vec{v}_1, \dots, \vec{v}_p$ in $\mathbb{R}^n$ and the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}. \tag{1}$$

- If (1) has only the trivial solution, the list of vectors is linearly independent.
- If (1) has a nontrivial solution, the list of vectors is linearly dependent.

Put another way, the list of vectors is linearly dependent if there exist some scalars $c_1, \dots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}. \tag{2}$$

(2) is called a linear dependence relation.

Example

Let

$$\vec{v}_1 = (1, 2, -1), \quad \vec{v}_2 = (0, 1, 1), \quad \vec{v}_3 = (1, 0, -3).$$

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent? If not, find a linear dependence relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Solution: We are interested in solutions of

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

so we row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This homogeneous system has 1 free variable.
So there are nontrivial solutions.
Thus, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not linearly independent (they are linearly dependent).

Example (cont.)

Any nontrivial solution gives a linear dependence relation.

Our general solution was

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{rcl} x_1 & = & -x_3 \\ x_2 & = & 2x_3 \\ x_3 & & \text{free} \end{array}$$

We pick any nonzero value for the free variable.

E.g. take $x_3 = 1 \implies x_1 = -1, \quad x_2 = 2.$

So

$$-(1,2,-1) + 2(0,1,1) + (1,0,-3) = (0,0,0)$$

is a linear dependence relation.

Check your answer!

Linear dependence of columns of a matrix

Problem: We're given a matrix A and asked whether the columns are linearly independent or dependent.

How to solve it: If

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$$

then we want to know if

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$$

has a nontrivial solution. This is the same as asking if the matrix equation

$$A\vec{x} = \vec{0}$$

has a nontrivial solution.

So we row reduce $\left[\begin{array}{c|c} A & \vec{0} \end{array} \right]$ and look for free variables.

Example

Are the columns of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 2 & 4 \\ 2 & 2 & 11 \end{bmatrix}$$

linearly independent or dependent?

Solution: We need to know if the matrix equation

$$A\vec{x} = \vec{0}$$

has a nontrivial solution.

So we row reduce

$$\left[\begin{array}{c|c} A & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -1 & 2 & 4 & 0 \\ 2 & 2 & 11 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right]$$

There are no free variables.
So there are no nontrivial solutions (only the trivial solution).
Thus the columns are linearly independent.

Solving problems about linear dependence/independence

So if you're given a list of vectors

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$$

and you want to know if they're linearly dependent or independent, you:

1. Form the matrix
$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix}$$
with these vectors as its columns.
2. Row reduce $\left[\begin{array}{c|c} A & \vec{0} \end{array} \right]$.
3. Then
   - if there are free variables, the vectors are dependent, and
   - if there are no free variables, the vectors are independent.
4. If they are linearly dependent and you want a linear dependence relation, find any nontrivial solution by picking any values for the free variables (not all zero).

Linear independence of single vector set

Question

When is the list $\vec{v}$ consisting of a single vector linearly independent?

Answer

The equation

$$x_1 \vec{v} = \vec{0}$$

has a nontrivial solution (i.e. a solution with $x_1 \neq 0$) iff $\vec{v} = \vec{0}$.

So $\vec{v}$ is linearly independent iff $\vec{v} \neq \vec{0}$.

Linear independence of a two vector set

Question

When is a list $\vec{v}_1, \vec{v}_2$ of two vectors linearly independent?

Answer

The set is linearly dependent iff

$$x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}$$

has a nontrivial solution – that is, a solution with $x_1 \neq 0$ and/or $x_2 \neq 0$.

- If $x_1 \neq 0$, then $\vec{v}_1 = -\frac{x_2}{x_1}\vec{v}_2$.
- If $x_2 \neq 0$, then $\vec{v}_2 = -\frac{x_1}{x_2}\vec{v}_1$.

So a list $\vec{v}_1, \vec{v}_2$ of two vectors is linearly dependent iff at least one of the vectors is a multiple of the other (i.e. they are parallel).

Example

If

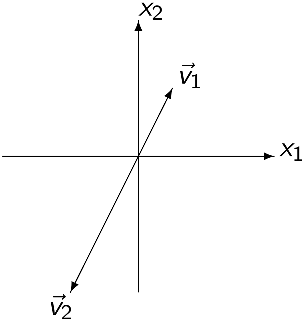
$$\vec{v}_1 = (1, 2) \qquad \text{and} \qquad \vec{v}_2 = (-2, -4),$$

is the list $\vec{v}_1, \vec{v}_2$ linearly independent or dependent?

Solution: Since

$$\vec{v}_2 = -2\vec{v}_1,$$

the list is linearly dependent.



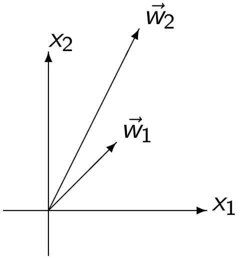
Example

If

$$\vec{w}_1 = (3,3) \qquad \text{and} \qquad \vec{w}_2 = (4,8),$$

is the list $\vec{w}_1, \vec{w}_2$ linearly independent or dependent?

Solution: Neither of these two vectors is a multiple of the other. Thus, they are linearly independent.



Characterization of linearly independent sets

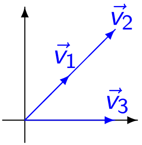
Theorem

A list $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ of two or more vectors is linearly dependent if and only if at least one of the vectors in the list is a linear combination of the others.

In fact, if the list is linearly dependent and $\vec{v}_1 \neq \vec{0}$, then some $\vec{v}_j$ $(j > 1)$ is a linear combination of the preceding vectors $\vec{v}_1, \dots, \vec{v}_{j-1}$.

Warning: The theorem does not say that in a linearly dependent set, every vector is a linear combination of the others.

Example:



$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent

- $\vec{v}_1$ is a lin comb of $\vec{v}_2$ and $\vec{v}_3$
- $\vec{v}_2$ is a lin comb of $\vec{v}_1$ and $\vec{v}_3$
- $\vec{v}_3$ is not a lin comb of $\vec{v}_1, \vec{v}_2$

Why is the theorem true?

Suppose you have a list of vectors:

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$$

If one vector is a linear combination of the others:

$$\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1} + c_{j+1}\vec{v}_{j+1} + \cdots + c_p\vec{v}_p$$

then

$$c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j + c_{j+1}\vec{v}_{j+1} + \cdots + c_p\vec{v}_p = \vec{0}$$

and so the set is linearly dependent.

If the set is linearly dependent, there is a linear dependence relation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$$

with at least one nonzero coefficient (say $c_j \neq 0$). Then

$$\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \cdots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1} - \frac{c_{j+1}}{c_j}\vec{v}_{j+1} - \cdots - \frac{c_p}{c_j}\vec{v}_p$$

and so one vector is a linear combination of the others.

Example

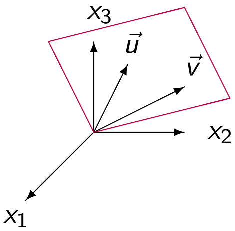
Suppose $\vec{u} = (0, 1, 2)$ and $\vec{v} = (1, 4, 2)$.

Then $\vec{u}, \vec{v}$ are linearly independent (neither is a multiple of the other).

So

$$\mathrm{Span}\{\vec{u}, \vec{v}\} = \{c_1\vec{u} + c_2\vec{v} \mid c_1, c_2 \in \mathbb{R}\}$$

is a plane through the origin.



Suppose we have a third vector $\vec{w}$. What is the geometric meaning of

$$\vec{u}, \vec{v}, \vec{w}$$

being linearly dependent or linearly independent?

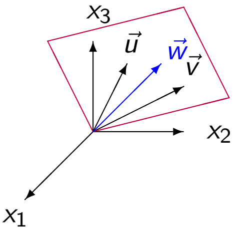
Example (cont.)

By our theorem, $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent iff one vector is a linear combination of the previous ones.

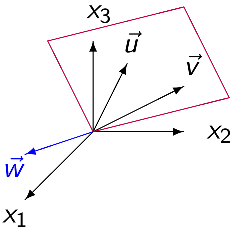
But $\vec{v}$ is not a linear combination of $\vec{u}$.

Thus

- $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent if $\vec{w}$ lies in the plane $\text{Span}\{\vec{u}, \vec{v}\}$.
- $\vec{u}, \vec{v}, \vec{w}$ are linearly independent if $\vec{w}$ does not lie in the plane $\text{Span}\{\vec{u}, \vec{v}\}$.



Linearly dependent



Linearly independent

Theorem

If

- $\vec{v}_1, \ldots, \vec{v}_p$ is a list of vectors in $\mathbb{R}^n$, and
- $p > n$

then the set is linearly dependent.

Reason

Arrange the vectors into a matrix:

$$A = [\vec{v}_1 \ \cdots \ \vec{v}_p] \qquad (A \text{ is } n \times p)$$

- Since $p > n$, the equation $A\vec{x} = \vec{0}$ has more variables than equations.
- So it has free variables.
- Thus, it has a nontrivial solution.
- Therefore, the columns are linearly dependent.

Note: This does not mean that if $p \leq n$, the set is linearly independent! It could be dependent or independent!

Example

Are the vectors

$$(1, 2, 3, -1), (0, 0, 1, 3), (1, 2, -1, 5), (1, 1, 1, 1), (0, 1, 0, 1)$$

linearly dependent or independent?

Solution: There are two ways to solve this problem.

1. We could form a matrix with these vectors as its columns, row reduce, and see if the system has a nontrivial solution (i.e. see if it has free variables).
2. We could use the previous theorem to say right away that it is linearly dependent because it consists of 5 vectors in $\mathbb{R}^4$ and $5 > 4$.

Moral of the story: If you have more vectors than the dimension of the space, they must be linearly dependent!

Theorem

If a list of vectors contains the zero vector, the vectors are linearly dependent.

Reason

- Suppose our list is
$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p.$$
- If necessary, renumber so that $\vec{v}_1 = \vec{0}$.
- Then
$$1\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p = \vec{0}$$
is a linear dependence relation.
- So the vectors are linearly dependent.

Most important point

If you're given a list of vectors

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$$

and you want to know if they're linearly dependent or independent, you:

1. Form the matrix
$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix}$$
with these vectors as its columns.
2. Row reduce $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$.
3. Then
   - if there are free variables, the vectors are dependent, and
   - if there are no free variables, the vectors are independent.
4. If they are linearly dependent and you want a linear dependence relation, find any nontrivial solution by picking any values for the free variables (not all zero).

This always works. The other shortcuts we discussed only work in special cases. Students often misuse them (use them when they don't actually apply) so be careful. When in doubt use the above procedure.

Midterm: Next class (Friday)

Logistics

- In class – please arrive early!!
- Bring your student ID!!
- Know your DGD number.
- No notes, books, calculators, or scrap paper.
- You will be asked to place your bags/jackets at the side/front/back of the room. Turn off cell phones!
- Write in pen.
- You may not leave in the final 10 minutes of the exam.

Material

- Covers up to and including last lecture (Lectures 1–6).
- 6 questions.
- Best way to study is to do all the recommended exercises.

Midterm

"Existence" questions: (EF)

- Is a linear system consistent or inconsistent?
- Is a vector equation consistent or inconsistent?
- Is a matrix equation consistent or inconsistent?
- Is one vector a linear combination of some others?
- Is one vector in the span of some others?

"Solution set" questions: (RREF)

- Find the general solution of a linear system.
- Find the general solution of a vector equation (answer in vector parametric notation).
- Find the general solution of a matrix equation (answer in vector parametric notation).
- Solutions set of a system and the corresponding homogeneous system.

Midterm

Other questions:

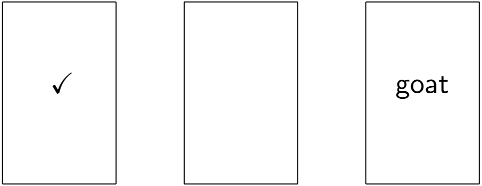
- Network flow.
- Vector parametric equation of a line.
- Matrix-vector product.

Warning: These lists might not be complete. You're responsible for all the material we've covered in class (except for weekend problems).

Tips:

- Read the question before starting!! When you're done, read the question again to make sure you answered it.
- Slow down!! You waste more time finding/fixing errors than you do going slow in the first place.
- Check your answers (even if the question doesn't ask you to)!!

Weekend problem



- You're on a game show
- There are 3 closed doors
- Behind 2 doors are goats, behind the other is 1 million dollars
- You choose a door (it remains closed)
- The host opens a door you did not choose, showing you a goat
- You are given a choice: stick with your original choice or switch to the other unopened door
- What should you do? Does it make any difference?

Weekend problem – solution

Strategy 1: You stay with your original pick

- you only win if your first pick was correct
- this happens $1/3$ of the time

Strategy 1: You switch to the other door

- if your original pick was right ($1/3$ the time), you lose
- if your original pick was wrong ($2/3$ the time), then
  - you picked a goat
  - the host must open the door with the other goat
  - you then switch to the door with the million dollars and win

Therefore, you should always switch. It doubles your chances of winning.

Next Time

For next time: Read Sections MO, MM

- Operations on matrices (addition, multiplication).
- This will help us simplify some calculations involving linear systems.