

MAT 1302B – Mathematical Methods II

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These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Last time

- General solutions (with free variables) to matrix (or vector) equations.
- Wrote solution sets in **vector parametric form**.
- Saw that **homogeneous systems** play an important role in such descriptions.
- Discussed relation between the solution set of a LS and the corresponding homogeneous system.

Last time: Homogeneous linear systems

Definition

A LS is **homogeneous** if it can be written in the form

$$A\vec{x} = \vec{0}$$

where A is an $m \times n$ matrix and $\vec{0} \in \mathbb{R}^m$.

A homogenous system always has the solution $\vec{x} = \vec{0} \in \mathbb{R}^n$, called the **trivial solution**.

A **nontrivial solution** is a solution $\vec{x} \neq \vec{0}$.

Theorem

The homogeneous equation $A\vec{x} = \vec{0}$ has a nontrivial solution iff it has at least one free variable.

Last time: Solutions to homogeneous systems

Description in terms of Span

The solution set of a **homogeneous** system can always be written in the form

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$$

for some vectors $\vec{v}_1, \dots, \vec{v}_p$.

Theorem

If $A\vec{x} = \vec{b}$ is consistent and \vec{p} is **any** solution, then full the solution set is the set of all vectors of the form

$$\vec{w} = \vec{p} + \vec{v}_h$$

where \vec{v}_h is **any** solution of the corresponding homogeneous equation $A\vec{x} = \vec{0}$. (Note: \vec{p} is called a **particular solution**.)

Example

Suppose

$$A = \begin{bmatrix} 1 & -3 & -4 & -4 \\ -2 & 6 & 6 & 4 \\ -1 & 3 & 2 & 0 \\ 3 & -9 & -10 & -8 \end{bmatrix}.$$

Find the general solution to the homogeneous equation $A\vec{x} = \vec{0}$.

Solution:

Example (cont.)

We want our solution in **vector parametric form**.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$$

So the solution set is:

Example (cont.)

$$A = \begin{bmatrix} 1 & -3 & -4 & -4 \\ -2 & 6 & 6 & 4 \\ -1 & 3 & 2 & 0 \\ 3 & -9 & -10 & -8 \end{bmatrix}$$

Solution set to $A\vec{x} = \vec{0}$: _____

New Question: Let

$$\vec{p} = (0, 1, 0, 0), \quad \vec{b} = A\vec{p} = (-3, 6, 3, -9) \quad (\text{2nd column of } A).$$

What is the general solution to the nonhomogeneous equation $A\vec{x} = \vec{b}$?

Solution:

$$\vec{x} =$$

Question: Does the following vector equation have a solution?

$$x_1 \begin{bmatrix} 2 \\ -2 \\ -4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \\ 6 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 4 \\ -7 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ -3 \\ 17 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer:

Question: Does it have **nontrivial** solutions?

Solution:

Today: Overview

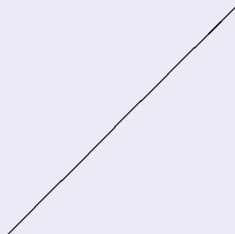
- Look at homogeneous systems in the language of vector equations.
- Examine an important notion: **linear dependence** of vectors.
- Later, we'll see that linear dependence is closely related to **dimension**.

Motivation: dimension

What is dimension?

Intuitive idea:

Line



Plane

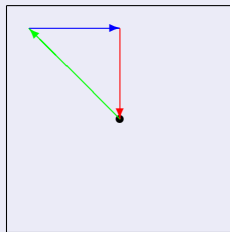
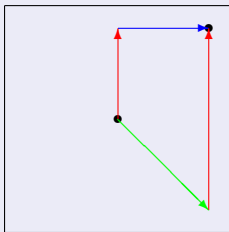
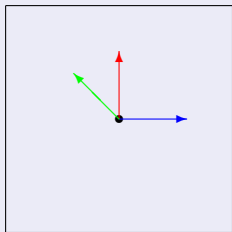


Space

Motivation: dimension

Problem

What about having three directions in a plane?



Resolution

- One direction is not needed (is redundant).
- With three vectors, there are multiple ways to get to the same place (even if we use a multiple of each vector only once).
- **Alternatively:** We can return to the origin, using (a multiple of) each vector only once.

Linear dependence

Definition

Consider a list of vectors $\vec{v}_1, \dots, \vec{v}_p$ in \mathbb{R}^n and the equation

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_p \vec{v}_p = \vec{0}. \quad (1)$$

- If (1) has only the trivial solution, the list of vectors is **linearly independent**.
- If (1) has a nontrivial solution, the list of vectors is **linearly dependent**.

Put another way, the list of vectors is linearly dependent if there exist some scalars c_1, \dots, c_p , not all zero, such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p = \vec{0}. \quad (2)$$

(2) is called a **linear dependence relation**.

Example

Let

$$\vec{v}_1 = (1, 2, -1), \quad \vec{v}_2 = (0, 1, 1), \quad \vec{v}_3 = (1, 0, -3).$$

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent? If not, find a linear dependence relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Solution: We are interested in solutions of

Example (cont.)

Linear dependence of columns of a matrix

Problem: We're given a matrix A and asked whether the columns are linearly independent or dependent.

How to solve it: If

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n]$$

then we want to know if

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$$

has a nontrivial solution. This is the same as asking if the matrix equation

$$A\vec{x} = \vec{0}$$

has a nontrivial solution.

So we row reduce $[A \mid \vec{0}]$ and look for free variables.

Example

Are the columns of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 2 & 4 \\ 2 & 2 & 11 \end{bmatrix}$$

linearly independent or dependent?

Solution:

Solving problems about linear dependence/independence

So if you're given a list of vectors

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$$

and you want to know if they're linearly dependent or independent, you:

- 1 Form the matrix

$$A = [\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_p]$$

with these vectors as its columns.

- 2 Row reduce $[A \mid \vec{0}]$.

- 3 Then

- ▶ if there are free variables, the vectors are dependent, and
- ▶ if there are no free variables, the vectors are independent.

- 4 If they are linearly dependent and you want a linear dependence relation, find **any nontrivial** solution by picking any values for the free variables (not all zero).

Linear independence of single vector set

Question

When is the list \vec{v} consisting of a single vector linearly independent?

Answer

Linear independence of a two vector set

Question

When is a list \vec{v}_1, \vec{v}_2 of two vectors linearly independent?

Answer

The set is linearly dependent iff

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{0}$$

has a nontrivial solution – that is, a solution with $x_1 \neq 0$ and/or $x_2 \neq 0$.

- If $x_1 \neq 0$, then $\vec{v}_1 = -\frac{x_2}{x_1} \vec{v}_2$.
- If $x_2 \neq 0$, then $\vec{v}_2 = -\frac{x_1}{x_2} \vec{v}_1$.

Example

If

$$\vec{v}_1 = (1, 2) \quad \text{and} \quad \vec{v}_2 = (-2, -4),$$

is the list \vec{v}_1, \vec{v}_2 linearly independent or dependent?

Solution:

Example

If

$$\vec{w}_1 = (3, 3) \quad \text{and} \quad \vec{w}_2 = (4, 8),$$

is the list \vec{w}_1, \vec{w}_2 linearly independent or dependent?

Solution:

Characterization of linearly independent sets

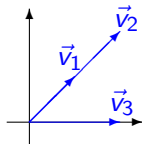
Theorem

A list $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ of two or more vectors is linearly dependent if and only if at least one of the vectors in the list is a linear combination of the others.

In fact, if the list is linearly dependent and $\vec{v}_1 \neq \vec{0}$, then some \vec{v}_j ($j > 1$) is a linear combination of the preceding vectors $\vec{v}_1, \dots, \vec{v}_{j-1}$.

Warning: The theorem does **not** say that in a linearly dependent set, **every** vector is a linear combination of the others.

Example:



$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent

- \vec{v}_1 is a lin comb of \vec{v}_2 and \vec{v}_3
- \vec{v}_2 is a lin comb of \vec{v}_1 and \vec{v}_3
- \vec{v}_3 is **not** a lin comb of \vec{v}_1, \vec{v}_2

Why is the theorem true?

Suppose you have a list of vectors:

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$$

If one vector is a linear combination of the others:

$$\vec{v}_j = c_1 \vec{v}_1 + \dots + c_{j-1} \vec{v}_{j-1} + c_{j+1} \vec{v}_{j+1} + \dots + c_p \vec{v}_p$$

then

$$c_1 \vec{v}_1 + \dots + c_{j-1} \vec{v}_{j-1} - \vec{v}_j + c_{j+1} \vec{v}_{j+1} + \dots + c_p \vec{v}_p = \vec{0}$$

and so the set is linearly dependent.

If the set is linearly dependent, there is a linear dependence relation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p = \vec{0}$$

with at least one nonzero coefficient (say $c_j \neq 0$). Then

$$\vec{v}_j = -\frac{c_1}{c_j} \vec{v}_1 - \dots - \frac{c_{j-1}}{c_j} \vec{v}_{j-1} - \frac{c_{j+1}}{c_j} \vec{v}_{j+1} - \dots - \frac{c_p}{c_j} \vec{v}_p$$

and so one vector is a linear combination of the others.

Example

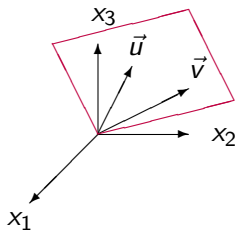
Suppose $\vec{u} = (0, 1, 2)$ and $\vec{v} = (1, 4, 2)$.

Then \vec{u}, \vec{v} are linearly independent (neither is a multiple of the other).

So

$$\text{Span}\{\vec{u}, \vec{v}\} = \{c_1\vec{u} + c_2\vec{v} \mid c_1, c_2 \in \mathbb{R}\}$$

is a plane through the origin.



Suppose we have a third vector \vec{w} . What is the geometric meaning of

$$\vec{u}, \vec{v}, \vec{w}$$

being linearly dependent or linearly independent?

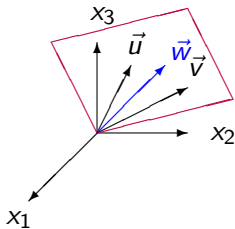
Example (cont.)

By our theorem, \vec{u} , \vec{v} , \vec{w} are linearly dependent iff one vector is a linear combination of the previous ones.

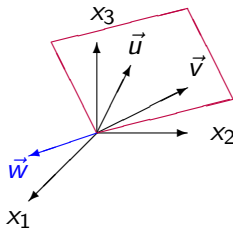
But \vec{v} is not a linear combination of \vec{u} .

Thus

- \vec{u} , \vec{v} , \vec{w} are linearly dependent if \vec{w} lies in the plane $\text{Span}\{\vec{u}, \vec{v}\}$.
- \vec{u} , \vec{v} , \vec{w} are linearly independent if \vec{w} does **not** lie in the plane $\text{Span}\{\vec{u}, \vec{v}\}$.



Linearly dependent



Linearly independent

Theorem

If

- $\vec{v}_1, \dots, \vec{v}_p$ is a list of vectors in \mathbb{R}^n , and
- $p > n$

then the set is linearly dependent.

Reason

Arrange the vectors into a matrix:

$$A = [\vec{v}_1 \ \cdots \ \vec{v}_p] \quad (A \text{ is } n \times p)$$

- Since $p > n$, the equation $A\vec{x} = \vec{0}$ has more variables than equations.
- So it has free variables.
- Thus, it has a nontrivial solution.
- Therefore, the columns are linearly dependent.

Note: This does **not** mean that if $p \leq n$, the set is linearly independent! It could be dependent or independent!

Example

Are the vectors

$$(1, 2, 3, -1), (0, 0, 1, 3), (1, 2, -1, 5), (1, 1, 1, 1), (0, 1, 0, 1)$$

linearly dependent or independent?

Solution: There are two ways to solve this problem.

1

2

Moral of the story:

Theorem

If a list of vectors contains the zero vector, the vectors are linearly dependent.

Reason

- Suppose our list is

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p.$$

- If necessary, renumber so that $\vec{v}_1 = \vec{0}$.
- Then

is a linear dependence relation.

- So the vectors are linearly dependent.

Most important point

If you're given a list of vectors

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$$

and you want to know if they're linearly dependent or independent, you:

- 1 Form the matrix

$$A = [\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_p]$$

with these vectors as its columns.

- 2 Row reduce $[A \mid \vec{0}]$.

- 3 Then

- ▶ if there are free variables, the vectors are dependent, and
- ▶ if there are no free variables, the vectors are independent.

- 4 If they are linearly dependent and you want a linear dependence relation, find **any nontrivial** solution by picking any values for the free variables (not all zero).

This **always** works. The other shortcuts we discussed only work in special cases. Students often misuse them (use them when they don't actually apply) so **be careful**. When in doubt use the above procedure.

Midterm: Next class (Friday)

Logistics

- In class – please arrive early!!
- Bring your student ID!!
- Know your **DGD number**.
- No notes, books, calculators, or scrap paper.
- You will be asked to place your bags/jackets at the side/front/back of the room. Turn off cell phones!
- Write in pen.
- You may not leave in the final 10 minutes of the exam.

Material

- Covers up to and including last lecture (Lectures 1–6).
- 6 questions.
- Best way to study is to do all the recommended exercises.

Midterm

“Existence” questions: (EF)

- Is a linear system consistent or inconsistent?
- Is a vector equation consistent or inconsistent?
- Is a matrix equation consistent or inconsistent?
- Is one vector a linear combination of some others?
- Is one vector in the span of some others?

“Solution set” questions: (RREF)

- Find the general solution of a linear system.
- Find the general solution of a vector equation (answer in vector parametric notation).
- Find the general solution of a matrix equation (answer in vector parametric notation).
- Solutions set of a system and the corresponding homogeneous system.

Midterm

Other questions:

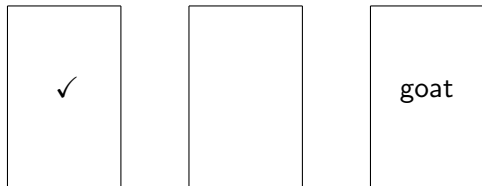
- Network flow.
- Vector parametric equation of a line.
- Matrix-vector product.

Warning: These lists might not be complete. You're responsible for all the material we've covered in class (except for weekend problems).

Tips:

- Read the question before starting!! When you're done, read the question again to make sure you answered it.
- Slow down!! You waste more time finding/fixing errors than you do going slow in the first place.
- Check your answers (even if the question doesn't ask you to)!!

Weekend problem



- You're on a game show
- There are 3 closed doors
- Behind 2 doors are goats, behind the other is 1 million dollars
- You choose a door (it remains closed)
- The host opens a door you did **not** choose, showing you a goat
- You are given a choice: stick with your original choice or switch to the other unopened door
- What should you do? Does it make any difference?

Weekend problem – solution

Next Time

For next time: Read Sections MO, MM

- Operations on matrices (addition, multiplication).
- This will help us simplify some calculations involving linear systems.