MAT 1302B – Mathematical Methods II

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Winter 2015 – Lecture 6

Announcements

First Midterm:

- Second class of next week.
- Covers Lectures 1–6 (up to and including today's lecture).
- Know your DGD number. You will be asked to write it on your exam.
- You must use pen, not pencil.
- No calculators.
- Bring your student ID.

DGDs:

- Next week TAs will go through some old/practice exams (available on course webpage).

Three languages: Solution sets

Theorem (How to translate)

If A is an $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$ and $\vec{b} \in \mathbb{R}^m$, then

1. the matrix equation $A\vec{x} = \vec{b}$,
2. the vector equation $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$, and
3. the LS with augmented matrix

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \tag{1}$$

all have the same solution set.

In the above,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

All of the above solution sets are found by row reducing the augmented matrix in (1).

Three languages: Existence of solutions

Theorem (How to translate)

Suppose

- A is an $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$, and
- $\vec{b} \in \mathbb{R}^m$.

then the following statements are equivalent:

1. The matrix equation $A\vec{x} = \vec{b}$ has a solution.
2. The vector equation $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$ has a solution.
3. $\vec{b}$ is a linear combination of $\vec{a}_1, \dots, \vec{a}_n$.
4. $\vec{b}$ is in $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$.
5. The LS with augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution.

So there are 5 ways of asking the same question! All are answered by row reducing the matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ to see if the rightmost column is a pivot column.

Equivalent questions: example

Suppose

$$\vec{a}_1 = (4, 7, -1), \quad \vec{a}_2 = (-3, 8, 5), \quad \vec{a}_3 = (2, 0, 1), \quad \vec{b} = (0, 1, -6).$$

Question 1: Is $\vec{b}$ a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$?

Question 2: Is $\vec{b}$ in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Question 3: Does the following vector equation have a solution?

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$$

Question 4: Does the matrix equation $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 4 & -3 & 2 \\ 7 & 8 & 0 \\ -1 & 5 & 1 \end{bmatrix}$ have a solution?

Question 5: Does the following linear system have a solution?

$$\begin{array}{rcrcrcr} 4x_1 & - & 3x_2 & + & 2x_3 & = & 0 \\ 7x_1 & + & 8x_2 & & & = & 1 \\ -x_1 & + & 5x_2 & + & x_3 & = & -6 \end{array}$$

Equivalent questions: example

We solve all of these questions by row reducing the matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$:

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] = \left[\begin{array}{ccc|c} 4 & -3 & 2 & 0 \\ 7 & 8 & 0 & 1 \\ -1 & 5 & 1 & -6 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 109/139 \\ 0 & 1 & 0 & -78/139 \\ 0 & 0 & 1 & -335/139 \end{array}\right]$$

Since the system is consistent, the answer to all the questions is YES.

Example

Suppose

$$\vec{a}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}.$$

If possible, express $\vec{b}$ as a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$.

Solution: We row reduce:

$$\left[\begin{array}{ccc|c} 3 & 0 & 3 & 6 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$\begin{array}{rcrcl} x_1 & & + & x_3 & = & 2 \\ & x_2 & - & x_3 & = & -1 \end{array} \Rightarrow \begin{array}{ll} x_1 & = 2 - x_3 \\ x_2 & = -1 + x_3 \\ x_3 & \text{free} \end{array}$$

Example (cont.)

General solution:

$$x_1 = 2 - x_3$$
$$x_2 = -1 + x_3$$
$$x_3 \text{ free}$$

Since we are asked to express $\vec{b}$ as a linear combination of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, any solution will do.

So we can pick any value for x_3 we like. Let's take $x_3 = 0$. Then

$$x_1 = 2, \quad x_2 = -1.$$

Then we have

$$2\vec{a}_1 - \vec{a}_2 + 0\vec{a}_3 = 2\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \vec{0} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} = \vec{b}$$

Other choices for x_3 give other answers.

Example

$$\vec{a}_1 = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

For which values of b_1, b_2, b_3 is $\vec{b} \in \operatorname{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Solution: We row reduce:

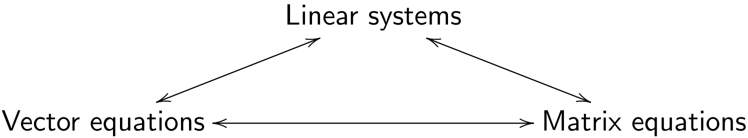
$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & b_1 \\ 2 & 1 & 1 & b_2 \\ 4 & 1 & 0 & b_3 \end{array}\right] \xrightarrow[2R_1+R_3]{R_1+R_2} \left[\begin{array}{ccc|c} -2 & 0 & 1 & b_1 \\ 0 & 1 & 2 & b_1 + b_2 \\ 0 & 1 & 2 & 2b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} -2 & 0 & 1 & b_1 \\ 0 & 1 & 2 & b_1 + b_2 \\ 0 & 0 & 0 & b_1 - b_2 + b_3 \end{array}\right]$$

The system is consistent if and only if $b_1 - b_2 + b_3 = 0$.
Therefore, $\vec{b} \in \operatorname{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ if and only if $b_1 - b_2 + b_3 = 0$.

Today: Overview

Three languages:



Up to now, our final descriptions of solution sets have been in the language of linear systems (at least for solution sets with free variables)

We want to complete this picture by describing solution sets in terms of vectors.

Homogeneous linear systems

Definition

A LS is homogeneous if it can be written in the form

$$A\vec{x} = \vec{0}$$

where A is an $m \times n$ matrix and $\vec{0} \in \mathbb{R}^m$.

A homogenous system always has the solution $\vec{x} = \vec{0} \in \mathbb{R}^n$, called the trivial solution.

A nontrivial solution is a solution $\vec{x} \neq \vec{0}$.

Theorem

The homogeneous equation $A\vec{x} = \vec{0}$ has a nontrivial solution iff it has at least one free variable.

Reason: Every LS has 0, 1 or infinitely many solutions!

Example

The linear system

$$\begin{array}{rcrcrcrl} 3x_1 & + & -2x_2 & & & + & 4x_4 & = 0 \\ -x_1 & & & + & 10x_3 & - & 8x_4 & = 0 \\ & & x_2 & - & x_3 & + & x_4 & = 0 \end{array}$$

is homogenous. This system is consistent (it has at least the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$). To find out if it has nontrivial solutions, we'd have to row reduce and see if there are free variables.

Example

$$\begin{array}{rcrcrcrl} 3x_1 & + & -2x_2 & & & + & 4x_4 & = 0 \\ -x_1 & & & + & 10x_3 & - & 8x_4 & = 4 \\ & & x_2 & - & x_3 & + & x_4 & = -2 \end{array}$$

is not a homogenous system. To know if this system is consistent or inconsistent, we would have to row reduce.

Example

Consider the linear system:

$$\begin{array}{rcrcrcl} -2x_1 & + & 6x_2 & - & x_3 & = & 0 \\ 4x_1 & + & 3x_2 & + & 2x_3 & = & 0 \\ 6x_1 & - & 5x_2 & + & 3x_3 & = & 0 \end{array}$$

We write down the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} -2 & 6 & -1 & 0 \\ 4 & 3 & 2 & 0 \\ 6 & -5 & 3 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the general solution is

$$\begin{array}{ccccccc} x_1 & & & + & \frac{1}{2}x_3 & = & 0 \\ & & x_2 & & & = & 0 \\ & & & & 0 & = & 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 = \dfrac{-1}{2}x_3 \\ x_2 = 0 \\ x_3 \quad \text{free} \end{array}$$

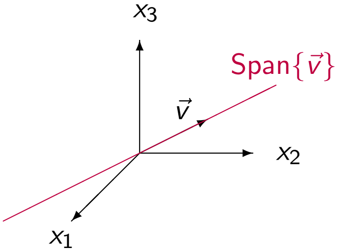
Since there are free variables, the system has nontrivial solutions.

Example (cont.)

So the general solution to $A\vec{x} = \vec{0}$, where A is the coefficient matrix of the above LS has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = x_3\vec{v}, \quad \text{where } \vec{v} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

This is the vector parametric form of the solution.



The solution set is $\text{Span}\{\vec{v}\}$, the line through $\vec{0}$ and $\vec{v}$. It consists of all scalar multiples of $\vec{v}$. The trivial solution corresponds to the zero multiple ($x_3 = 0$).

Another example

Consider the equation (also a LS)

$$3x_1 - 2x_2 - 5x_3 = 0$$

The solution set is given by

$$x_1 = \frac{2}{3}x_2 + \frac{5}{3}x_3$$
$$x_2,\ x_3 \quad \text{free}$$

So the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 + \frac{5}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{5}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{5}{3} \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

vector parametric form

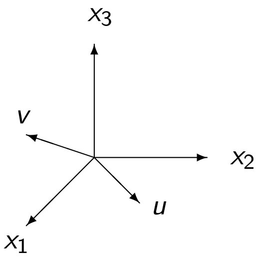
Another example (cont.)

So the general solution is

$$x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{5}{3} \\ 0 \\ 1 \end{bmatrix} = x_2 \vec{u} + x_3 \vec{v}, \quad \text{where} \quad \vec{u} = \left(\frac{2}{3}, 1, 0 \right), \quad \vec{v} = \left(\frac{5}{3}, 0, 1 \right)$$

where x_2, x_3 can be any scalars.

So the solution set is $\mathrm{Span}\{\vec{u}, \vec{v}\}$



Solutions to homogeneous systems

Description in terms of Span

The solution set of a homogeneous system can always be written in the form

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$$

for some vectors $\vec{v}_1, \ldots, \vec{v}_p$.

To find the set of vectors $\{v_1, \ldots, v_p\}$ we:

- solve the linear system as usual,
- write the solution in vector notation, (replace basic variables by their expressions in terms of free variables),
- split the vector into a sum, one term for each free variable, and factor out the free variables – the vectors appearing yield the set.

E.g.

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \vdots \end{bmatrix} + x_4 \begin{bmatrix} \vdots \end{bmatrix} \quad \text{(if } x_1 \text{ and } x_4 \text{ are the free variables)}$$

Solutions to homogeneous systems

Important note: The above is only true for homogeneous systems! For example, the system

$$x = 1$$
$$y = 2$$

has the unique solution $(x, y) = (1, 2)$ and this solution set is not the span of a set of vectors (for instance, $\vec{0}$ is always in any span).

Another note: If a homogenous system has *only* the trivial solution, the solution set is $\mathrm{Span}\{\vec{0}\}$.

Nonhomogeneous systems

Example: Describe all solutions of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & -6 \\ 2 & 1 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}.$$

Solution: We first row reduce:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ -1 & 3 & -6 & 5 \\ 2 & 1 & 5 & 4 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then we write the general solution:

$$\begin{array}{rcrcl} x_1 & & +\ 3x_3 & = & 1 \\ & x_2 & -\ \ x_3 & = & 2 \\ & & 0 & = & 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} x_1 & = 1 - 3x_3 \\ x_2 & = 2 + x_3 \\ x_3 & \text{free} \end{array}$$

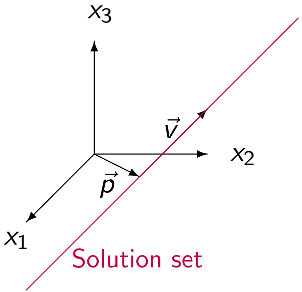
Then we switch to vector notation:

$$\begin{array}{lcl} x_1 & = & 1-3x_3 \\ x_2 & = & 2+x_3 \\ x_3 & & \text{free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-3x_3 \\ 2+x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

So the vector parametric form of the general solution is

$$\vec{x} = \vec{p} + t\vec{v}, \quad t \in \mathbb{R}$$

where $\vec{p} = (1, 2, 0)$ and $\vec{v} = (-3, 1, 1)$.
Note: We have replaced x_3 by a parameter t.



The solution set is a line through $\vec{p}$, parallel to $\vec{v}$.

Homogeneous and nonhomogeneous systems

Question

What is the relationship between homogeneous and nonhomogeneous systems?

In our previous example, we solved $A\vec{x} = \vec{b}$ and described the solution set as

$$\vec{x} = \vec{p} + t\vec{v}, \quad t \in \mathbb{R}.$$

If we solved the corresponding homogeneous system

$$A\vec{x} = \vec{0} \qquad \text{(same } A \text{ as above)}$$

we would get the solution

$$\vec{x} = t\vec{v}, \quad t \in \mathbb{R} \qquad \text{(same } \vec{v} \text{ as above).}$$

Note: The only difference is the presence of the vector $\vec{p}$, which is a solution of the nonhomogeneous system (corresponding to parameter value $t = 0$).

Homogenous and nonhomogeneous systems

Theorem

If $A\vec{x} = \vec{b}$ is consistent and $\vec{p}$ is any solution, then full the solution set is the set of all vectors of the form

$$\vec{w} = \vec{p} + \vec{v}_h$$

where $\vec{v}_h$ is any solution of the corresponding homogeneous equation $A\vec{x} = \vec{0}$.

Notation

In the above, $\vec{p}$ is called a particular solution.

So if we know all the solutions to the homogenous system (obtained by setting the constant terms to zero), then we get all the solutions to the nonhomogeneous system just by adding any particular solution $\vec{p}$ to them all!

Advantages

The relation between homogeneous and nonhomogeneous systems can save us a lot of time.

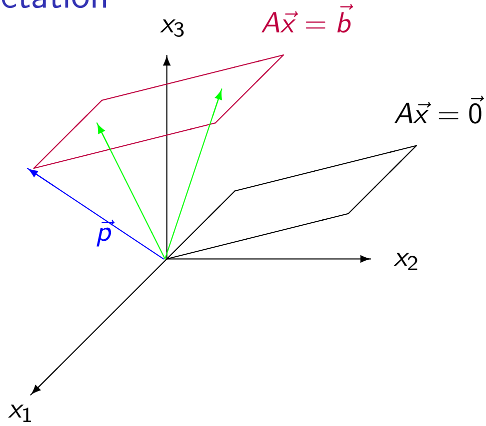
Question

Suppose we have to solve a bunch of systems with the same coefficients but different constant terms. How could we do this efficiently?

Answer

1. Solve the corresponding homogenous system.
2. For each system find one particular solution (sometimes this is easy to do by inspection).
3. Add the particular solutions to the solution set of the homogenous system.

Geometric interpretation



The solution set of $A\vec{x} = \vec{b}$ is obtained by shifting the solution set of $A\vec{x} = \vec{0}$ by a particular solution $\vec{p}$ (i.e. vector satisfying $A\vec{p} = \vec{b}$).

Note: We can shift by any particular solution (green arrows represent other choices).

Another example

Find the solution set to the matrix equation

$$\begin{bmatrix} 1 & -2 & -1 & 2 \\ -1 & 2 & 3 & 4 \\ -2 & 4 & 0 & -10 \end{bmatrix} \vec{x} = \begin{bmatrix} -5 \\ 13 \\ 2 \end{bmatrix}.$$

Write the answer in vector parametric form.

Solution:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 2 & -5 \\ -1 & 2 & 3 & 4 & 13 \\ -2 & 4 & 0 & -10 & 2 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 5 & -1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$\begin{aligned} x_1 &= 2x_2 - 5x_4 - 1 \\ x_3 &= -3x_4 + 4 \\ x_2, x_4 &\quad \text{free} \end{aligned}$$

We switch to vector notation:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_4 - 1 \\ x_2 \\ -3x_4 + 4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}$$

Question: What is the solution set to the corresponding homogeneous equation

$$\begin{bmatrix} 1 & -2 & -1 & 2 \\ -1 & 2 & 3 & 4 \\ -2 & 4 & 0 & -10 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

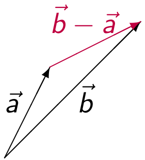
(same equation but with zeros on the right)?

Answer: We just remove the particular solution. The solution set is

$$\operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

How to find the vector equation of a line

Question: What is the vector with tail at point $\vec{a}$ and tip at point $\vec{b}$?



Example

Vector with tip at point $(3, -1, 5)$ and tail at $(-2, 0, 8)$ is

$$\vec{v} = (3 - (-2), -1 - 0, 5 - 8) = (5, -1, -3).$$

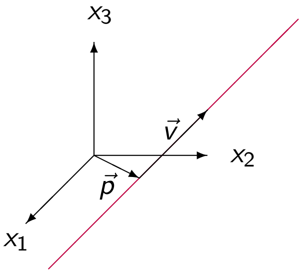
How to find the vector equation of a line

Procedure

To find the vector equation of a line

1. Find a vector $\vec{v}$ parallel to the line (can use any two points on the line and find the vector from one to the other as above).
2. Find a point $\vec{p}$ on the line.
3. The vector equation is then

$$\vec{p} + t\vec{v}, \quad t \in \mathbb{R}.$$



Example

Question: Find the vector equation of the line through the points $(2,0,3)$ and $(-1,1,2)$.

Answer: A vector parallel to the line is

$$\vec{v} = (2,0,3) - (-1,1,2) = (3,-1,1).$$

A point on the line is

$$\vec{p} = (2,0,3).$$

Therefore, the vector equation is

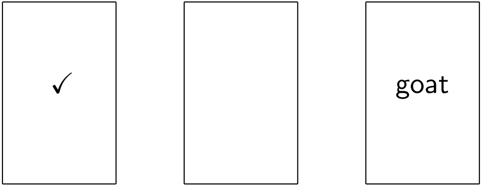
$$(2,0,3) + t(3,-1,1), \quad t \in \mathbb{R}.$$

Note: There are lots of vector equations of any given line.

Example: Another vector equation of the above line is

$$(-1,1,2) + t(3,-1,1), \quad t \in \mathbb{R}.$$

Weekend problem



- You're on a game show
- There are 3 closed doors
- Behind 2 doors are goats, behind the other is 1 million dollars
- You choose a door (it remains closed)
- The host opens a door you did not choose, showing you a goat
- You are given a choice: stick with your original choice or switch to the other unopened door
- What should you do? Does it make any difference?

Next time

For next time: Read Section LI

- Discuss homogeneous equations in the language of vector equations
- Linear dependence/independence

Important: Students often find linear dependence a difficult topic. So **read the section before class!**