MAT 1302B – Mathematical Methods II

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Announcements

First midterm: In class on Friday, February 6. Calculators are not allowed.

Old midterms: Posted on the course webpage.

Warning! Each year, the midterms cover slightly different material. So DON'T assume that this year's midterms will cover the same material as midterms from previous years.

The first midterm will cover Lectures 1–6.

Last time

New language: We introduced the language of vectors and vector equations.

Important: Once we get past the new notation/terminology, we see that questions in the language of vector equations can be translated into questions about linear systems.

What you need to know: Learn the terminology and how to translate between the languages of vector equations and linear systems.

Last time

- vector: matrix with one column
- $\mathbb{R}^n$: the set of all $n \times 1$ matrices
- plotting points and drawing vectors in $\mathbb{R}^n$, $n \leq 3$
- operations:
  - vector addition
  - scalar multiplication
- linear combinations, span
- relation between linear combinations and linear systems

Last time: Linear combinations

Definition (linear combination)

For $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_k$, the vector

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

is called a linear combination of $\vec{v}_1, \dots, \vec{v}_k$ with weights (or coefficients) $c_1, \dots, c_k$.

Example

$$\vec{v}_1 - \vec{v}_2, \quad \vec{v}_1 \quad (= 1\vec{v}_1 + 0\vec{v}_2), \quad \vec{0} \quad (= 0\vec{v}_1 + 0\vec{v}_2)$$

are linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

Last time: Vector equations and linear systems

Theorem (how to translate)

A vector equation
$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_k\vec{a}_k = \vec{b}$$
has the same solution set as the LS with augmented matrix
$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k & \vec{b} \end{array}\right] \tag{1}$$
In particular, $\vec{b}$ can be written as a linear combination of $\vec{a}_1, \dots, \vec{a}_k$ if and only if there exists a solution to the LS corresponding to (1).

Definition (Span)

If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_k$ is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \dots, \vec{v}_k$. It is denoted
$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \quad \text{or} \quad \langle \vec{v}_1, \dots, \vec{v}_k \rangle.$$

Example

Suppose

$$\vec{a}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 10 \\ -5 \\ 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 27 \\ -4 \\ 12 \end{bmatrix}.$$

Is $\vec{b}$ in $\mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Solution: We want to know if the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$$

has a solution.

$$\left[\begin{array}{ccc|c} 3 & 2 & 10 & 27 \\ -1 & 1 & -5 & -4 \\ 1 & 1 & 3 & 12 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 7 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The rightmost column is a pivot column and so the answer is NO.

Today

- Develop further the language of vector (and matrix) equations.
- As above, we will see that such equations are closely related to linear systems.
- New language will allow us to simplify some calculations and attack problems in a different way.
- Important: Concentrate on learning the definitions/terminology and how to translate between the languages.

Matrix equations

Definition (matrix-vector multiplication)

Suppose

- A is an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$
- $\vec{x} \in \mathbb{R}^n$

Then we define

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \stackrel{\text{def}}{=} x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

Example 1

$$\begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

Note: The number of entries in the result (the vector on the right hand side) is the same as the number of rows in A.

Example 2

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 5 & 4 & 2 \\ 0 & -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 0 \end{bmatrix}$$

Note again: The number of entries in the result (the vector on the right hand side) is the same as the number of rows in the matrix. This will always happen.

Example 3

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & \pi & 8 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -7 \\ 3 \end{bmatrix}$$

Not defined since the number of columns in the matrix is not equal to the number of entries (rows) in the vector.

Shortcut

$$\begin{bmatrix}1 & 0 & -1 & 0\\ 2 & 5 & 4 & 2\\ 0 & -1 & 2 & 4\end{bmatrix}\begin{bmatrix}1\\0\\2\\-1\end{bmatrix} = 1\begin{bmatrix}1\\2\\0\end{bmatrix} + 0\begin{bmatrix}0\\5\\-1\end{bmatrix} + 2\begin{bmatrix}-1\\4\\2\end{bmatrix} + (-1)\begin{bmatrix}0\\2\\4\end{bmatrix}$$

$$= \begin{bmatrix}1\cdot 1 + 0\cdot 0 + (-1)2 + 0(-1)\\ 2\cdot 1 + 5\cdot 0 + 4\cdot 2 + 2(-1)\\ 0\cdot 1 + (-1)0 + 2\cdot 2 + 4(-1)\end{bmatrix} = \begin{bmatrix}-1\\8\\0\end{bmatrix}$$

We can skip the middle step.

Example:

$$\begin{bmatrix}2 & 1 & 0\\ -1 & 3 & 2\\ 2 & 4 & 0\end{bmatrix}\begin{bmatrix}0\\-1\\2\end{bmatrix} = \begin{bmatrix}2\cdot 0 + 1(-1) + 0\cdot 2\\ (-1)0 + 3(-1) + 2\cdot 2\\ 2\cdot 0 + 4(-1) + 0\cdot 2\end{bmatrix} = \begin{bmatrix}-1\\1\\-4\end{bmatrix}$$

Matrix-vector multiplication and linear combinations

Another example

Suppose $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \mathbb{R}^n$. Then

$$4\vec{u} - \vec{v} + \vec{w} - 8\vec{z} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{z} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 1 \\ -8 \end{bmatrix}$$

We see from the above that our matrix-vector multiplication is another way of writing linear combinations!!

So we can easily translate between matrix-vector multiplication and linear combinations.

Relation to linear systems

The linear system

$$\begin{array}{rcrcrcrcr} 2x_1 & & x_2 & + & 3x_3 & - & x_4 & = & 8 \\ & & x_2 & - & x_3 & + & 10x_4 & = & -4 \\ x_1 & & & + & x_3 & - & 2x_4 & = & 0 \end{array}$$

is equivalent to the vector equation

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$$

which is equivalent to the matrix equation

$$\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & -1 & 10 \\ 1 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$$

Note that the matrix on the left is the coefficient matrix of the linear system!

The matrix equation

$$\begin{bmatrix} 3 & 4 & 7 \\ 0 & -1 & -3 \\ 2 & 8 & -5 \\ 1 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 2 \\ -3 \\ 1 \end{bmatrix} \qquad \left(\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

is equivalent to the vector equation

$$x_1 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 8 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -3 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$

which is equivalent to the linear system

$$\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & 7x_3 & = & 10 \\ & & -x_2 & - & 3x_3 & = & 2 \\ 2x_1 & + & 8x_2 & - & 5x_3 & = & -3 \\ x_1 & & & & & = & 1 \end{array}$$

Three languages: Solution sets

Theorem (How to translate)

If A is an $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$ and $\vec{b} \in \mathbb{R}^m$, then

1. the matrix equation $A\vec{x} = \vec{b}$,
2. the vector equation $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$, and
3. the LS with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n & \vec{b} \end{array}\right] = \left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \tag{2}$$

all have the same solution set.

In the above,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

All of the above solution sets are found by row reducing the augmented matrix in (2).

Three languages: Existence of solutions

Theorem (How to translate)

Suppose

- A is an $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and
- $\vec{b} \in \mathbb{R}^m$.

then the following statements are equivalent:

1. The matrix equation $A\vec{x} = \vec{b}$ has a solution.
2. The vector equation $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$ has a solution.
3. $\vec{b}$ is a linear combination of $\vec{a}_1, \ldots, \vec{a}_n$.
4. $\vec{b}$ is in $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$.
5. The LS with augmented matrix $\left[\begin{array}{ccc|c} \vec{a}_1 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$ has a solution.

So there are 5 ways of asking the same question! All are answered by row reducing the matrix $\left[\begin{array}{ccc|c} \vec{a}_1 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$ to see if the rightmost column is a pivot column.

Example

Suppose

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}.$$

Does the equation

$$A\vec{x} = \vec{b}$$

have a solution? If so, find the general solution.

Solution: By our translation theorem, we find the solution by row reducing

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 8 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

So the answer is YES. There is a unique solution:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

Check your answer!

We check by matrix-vector multiplication that $A\vec{x} = \vec{b}$.

$$A\vec{x} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 0 \cdot 0 + (-2)(-3) \\ 2 \cdot 2 + (-1)0 + 1(-3) \\ 3 \cdot 2 + 1 \cdot 0 + 1(-3) \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} = \vec{b} \quad \checkmark$$

Example

Question: Suppose

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

Is $\vec{b}$ a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$?

Answer: Write down the corresponding augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 1 & 4 & -5 & 2 \\ -2 & -6 & 8 & -2 \end{array}\right] \xrightarrow[2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & -4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The last column is a pivot column, so the answer is NO.

Question

How could we have asked essentially the same question in a different way?

Original question

Suppose

$$\vec{a}_1 = (1, 1, -2), \quad \vec{a}_2 = (3, 4, -6), \quad \vec{a}_3 = (-4, -5, 8), \quad \vec{b} = (2, 2, -2).$$

Is $\vec{b}$ a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$?

Variation 1: Does the vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ have a solution?

Variation 2: Is $\vec{b}$ in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Variation 3: Does the matrix equation $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 4 & -5 \\ -2 & -6 & 8 \end{bmatrix}$ have a solution?

Variation 4: Does the the following linear system have a solution?

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 4x_3 & = & 2 \\ x_1 & + & 4x_2 & - & 5x_3 & = & 2 \\ -2x_1 & - & 6x_2 & + & 8x_3 & = & -2 \end{array}$$

Question

For what matrices A does the equation

$$A\vec{x} = \vec{b}$$

have a solution, no matter what $\vec{b}$ is?

Answer: We solve the equation by row reducing

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$$

- This will have a solution when the rightmost column is not a pivot column.
- What ensures that the rightmost column will never be a pivot column?

If A has a pivot position in each row, then there will never be a pivot position in the rightmost column of $[A|\vec{b}]$!

Guaranteed existence of solutions

Theorem

Suppose A is an $m \times n$ matrix. Then the following statements are logically equivalent (i.e. they are either all true or all false).

1. For any $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2. A has a pivot position in every row.
3. Every $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
4. The columns of A span all of $\mathbb{R}^m$ (i.e. every $\vec{b} \in \mathbb{R}^m$ is in the span of the columns of A).

Example

Does the equation

$$\begin{bmatrix} 4 & -4 & 8 \\ 3 & -3 & 2 \end{bmatrix} \vec{x} = \vec{b}$$

have a solution for every $\vec{b} \in \mathbb{R}^2$?

Solution: We need to find out if the matrix has a pivot position in every row. So we row reduce.

$$\begin{bmatrix} 4 & -4 & 8 \\ 3 & -3 & 2 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

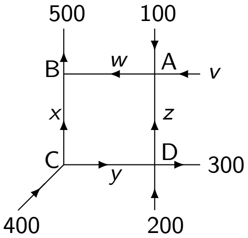
There is a pivot position in each row and so the answer is YES. Note that we row reduce the matrix A and not $[A|\vec{b}]$.

Equivalent conclusions:

1. Every $\vec{b} \in \mathbb{R}^2$ is a linear combination of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$, and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.
2. $$\text{Span}\left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \end{bmatrix} \right\} = \mathbb{R}^2$$

Application: Network flow

Suppose the traffic flow in a city is depicted by



At each intersection, the total flow in must equal the total flow out:

Intersection	**Flow in**		**Flow out**
A	$100 + v + z$	$=$	w
B	$x + w$	$=$	500
C	400	$=$	$x + y$
D	$y + 200$	$=$	$300 + z$

Also, total overall flow in must equal flow out:

$$400 + 200 + 100 + v = 300 + 500$$

We get the linear system

$$\begin{array}{rcrcrcrcrcr} v & - & w & & & & & + & z & = & -100 \\ & & w & + & x & & & & & = & 500 \\ & & & & x & + & y & & & = & 400 \\ & & & & & & y & - & z & = & 100 \\ v & & & & & & & & & = & 100 \end{array}$$

and we solve by row reduction.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & -100 \\ 0 & 1 & 1 & 0 & 0 & 500 \\ 0 & 0 & 1 & 1 & 0 & 400 \\ 0 & 0 & 0 & 1 & -1 & 100 \\ 1 & 0 & 0 & 0 & 0 & 100 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then we go back to equation form...

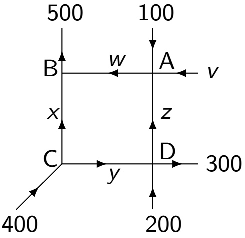
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} v &= 100 \\ w - z &= 200 \\ x + z &= 300 \\ y - z &= 100 \end{aligned} \Rightarrow \begin{aligned} v &= 100 \\ w &= 200 + z \\ x &= 300 - z \\ y &= 100 + z \end{aligned}$$

So v, w, x, y are basic variables and z is free.

Note: We must have $v, w, x, y, z \geq 0$. Thus, we must also have $z \leq 300$ (to ensure that $x \geq 0$).

So in our parametric solution, we limit the parameter z to have values between 0 and 300.

Solution:



$$v = 100$$
$$w = 200 + z$$
$$x = 300 - z$$
$$y = 100 + z$$
$$0 \leq z \leq 300$$

Additional questions:

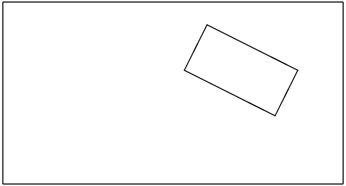
1. What are the minimum and maximum flows in the branch w?

   Answer: The minimum is 200 and the maximum is 500.

2. If the flow along z is restricted to a maximum of 150 due to road construction, what is the maximum flow along y?

   Answer: 250

Weekend problem (for fun)

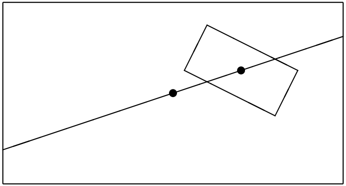


- You bake a delicious rectangular cake.
- In the middle of the night, your roommate gets hungry, sneaks into the kitchen, and cuts a rectangular piece out of your cake in a random position, at a random angle.

Question: How can you cut the cake in two, with a single straight cut, in such a way that each piece has the exact same amount of cake? You have only a knife and a straightedge (a ruler with no markings).

Note: There is one "cheating" answer and one "real" answer.

Weekend problem – Solution



Cheating answer: Cut the cake “sideways”, right in the middle.

Real answer: Cut the cake along the straight line joining the centres of the two rectangles.

- Any line through the centre of a rectangle cuts it in half.
- Therefore, the slice cuts the larger rectangle in half, plus the missing piece in half.
- So the remaining cake is cut exactly in half.

Note: You can find the centre of each rectangle – it's the point of intersection of the diagonals.

Next time

For next time: Read Sections HSE, LC

- Today we developed a new language (vector and matrix equations) for discussing problems equivalent to linear systems
- Up to now, our final descriptions of solution set have been in the language of linear systems
- Next time we will discuss how to describe solution sets in terms of vectors – this will give us new insight into the solution sets