

MAT 1302B – Mathematical Methods II

Alistair Savage

Mathematics and Statistics
University of Ottawa

Winter 2015 – Lecture 5

These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Announcements

First midterm: In class on Friday, February 6. Calculators are not allowed.

Old midterms: Posted on the course webpage.

Warning! Each year, the midterms cover slightly different material. So **DON'T** assume that this year's midterms will cover the same material as midterms from previous years.

The first midterm will cover Lectures 1–6.

Last time

New language: We introduced the language of vectors and vector equations.

Important: Once we get past the new notation/terminology, we see that questions in the language of vector equations can be translated into questions about linear systems.

What you need to know: Learn the terminology and how to translate between the languages of vector equations and linear systems.

Last time

- **vector**: matrix with one column
- \mathbb{R}^n : the set of all $n \times 1$ matrices
- plotting points and drawing vectors in \mathbb{R}^n , $n \leq 3$
- **operations**:
 - ▶ vector addition
 - ▶ scalar multiplication
- **linear combinations, span**
- relation between linear combinations and linear systems

Last time: Linear combinations

Definition (linear combination)

For $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$ and scalars c_1, c_2, \dots, c_k , the vector

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

is called a **linear combination** of $\vec{v}_1, \dots, \vec{v}_k$ with **weights** (or **coefficients**) c_1, \dots, c_k .

Example

$$\vec{v}_1 - \vec{v}_2, \quad \vec{v}_1 \quad (= 1\vec{v}_1 + 0\vec{v}_2), \quad \vec{0} \quad (= 0\vec{v}_1 + 0\vec{v}_2)$$

are linear combinations of \vec{v}_1 and \vec{v}_2 .

Last time: Vector equations and linear systems

Theorem (how to translate)

A vector equation

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_k \vec{a}_k = \vec{b}$$

has the same solution set as the LS with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k & \vec{b} \end{array} \right] \quad (1)$$

In particular, \vec{b} can be written as a linear combination of $\vec{a}_1, \dots, \vec{a}_k$ **if and only if** there exists a solution to the LS corresponding to (1).

Definition (Span)

If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_k$ is called the **subset of \mathbb{R}^n spanned (or generated)** by $\vec{v}_1, \dots, \vec{v}_k$. It is denoted

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \quad \text{or} \quad \langle \vec{v}_1, \dots, \vec{v}_k \rangle.$$

Example

Suppose

$$\vec{a}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 10 \\ -5 \\ 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 27 \\ -4 \\ 12 \end{bmatrix}.$$

Is \vec{b} in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Solution: We want to know if the vector equation

has a solution.

Today

- Develop further the language of vector (and matrix) equations.
- As above, we will see that such equations are closely related to linear systems.
- New language will allow us to simplify some calculations and attack problems in a different way.
- **Important:** Concentrate on learning the definitions/terminology and how to translate between the languages.

Matrix equations

Definition (matrix-vector multiplication)

Suppose

- A is an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$
- $\vec{x} \in \mathbb{R}^n$

Then we define

$$A\vec{x} = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n] \vec{x} = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \stackrel{\text{def}}{=} x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

Example 1

$$\begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} =$$

Note: The number of entries in the result (the vector on the right hand side) is _____

Example 2

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 5 & 4 & 2 \\ 0 & -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} =$$

Note again:

Example 3

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & \pi & 8 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -7 \\ 3 \end{bmatrix}$$

Shortcut

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 5 & 4 & 2 \\ 0 & -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} =$$

Example:

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \\ 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} =$$

Matrix-vector multiplication and linear combinations

Another example

Suppose $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \mathbb{R}^n$. Then

$$4\vec{u} - \vec{v} + \vec{w} - 8\vec{z} =$$

We see from the above that our matrix-vector multiplication is another way of writing linear combinations!!

So we can easily **translate** between matrix-vector multiplication and linear combinations.

Relation to linear systems

The linear system

$$\begin{array}{rccccrcr} 2x_1 & x_2 & + & 3x_3 & - & x_4 & = & 8 \\ & x_2 & - & x_3 & + & 10x_4 & = & -4 \\ x_1 & & + & x_3 & - & 2x_4 & = & 0 \end{array}$$

is equivalent to the vector equation

which is equivalent to the matrix equation

The matrix equation

$$\begin{bmatrix} 3 & 4 & 7 \\ 0 & -1 & -3 \\ 2 & 8 & -5 \\ 1 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 2 \\ -3 \\ 1 \end{bmatrix} \quad \left(\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

is equivalent to the vector equation

which is equivalent to the linear system

Three languages: Solution sets

Theorem (How to translate)

If A is an $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$ and $\vec{b} \in \mathbb{R}^m$, then

- 1 the matrix equation $A\vec{x} = \vec{b}$,
- 2 the vector equation $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$, and
- 3 the LS with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n & \vec{b} \end{array} \right] = \left[A \mid \vec{b} \right] \quad (2)$$

all have the same solution set.

In the above,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

All of the above solution sets are found by row reducing the augmented matrix in (2).

Three languages: Existence of solutions

Theorem (How to translate)

Suppose

- A is an $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$, and
- $\vec{b} \in \mathbb{R}^m$.

then the following statements are equivalent:

- 1 The matrix equation $A\vec{x} = \vec{b}$ has a solution.
- 2 The vector equation $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$ has a solution.
- 3 \vec{b} is a linear combination of $\vec{a}_1, \dots, \vec{a}_n$.
- 4 \vec{b} is in $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$.
- 5 The LS with augmented matrix $\left[\begin{array}{ccc|c} \vec{a}_1 & \dots & \vec{a}_n & \vec{b} \end{array} \right]$ has a solution.

So there are 5 ways of asking the same question! **All** are answered by row reducing the matrix $\left[\begin{array}{ccc|c} \vec{a}_1 & \dots & \vec{a}_n & \vec{b} \end{array} \right]$ to see if the rightmost column is a pivot column.

Example

Suppose

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}.$$

Does the equation

$$A\vec{x} = \vec{b}$$

have a solution? If so, find the general solution.

Solution:

So the answer is _____

Check your answer!

Example

Question: Suppose

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

Is \vec{b} a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$?

Answer:

Question

How could we have asked essentially the same question in a different way?

Original question

Suppose

$$\vec{a}_1 = (1, 1, -2), \quad \vec{a}_2 = (3, 4, -6), \quad \vec{a}_3 = (-4, -5, 8), \quad \vec{b} = (2, 2, -2).$$

Is \vec{b} a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$?

Variation 1:

Variation 2:

Variation 3:

Variation 4:

Question

For what matrices A does the equation

$$A\vec{x} = \vec{b}$$

have a solution, no matter what \vec{b} is?

Answer: We solve the equation by row reducing

- This will have a solution when



Guaranteed existence of solutions

Theorem

Suppose A is an $m \times n$ matrix. Then the following statements are logically equivalent (i.e. they are either all true or all false).

- 1 For any $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
- 2 A has a pivot position in every row.
- 3 Every $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A .
- 4 The columns of A span all of \mathbb{R}^m (i.e. every $\vec{b} \in \mathbb{R}^m$ is in the span of the columns of A).

Example

Does the equation

$$\begin{bmatrix} 4 & -4 & 8 \\ 3 & -3 & 2 \end{bmatrix} \vec{x} = \vec{b}$$

have a solution for every $\vec{b} \in \mathbb{R}^2$?

Solution: We need to find out if

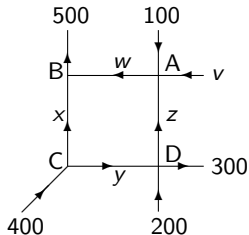
Equivalent conclusions:

1

2

Application: Network flow

Suppose the traffic flow in a city is depicted by



At each intersection, the total flow in must equal the total flow out:

Intersection	Flow in	Flow out
A		
B		
C		
D		

Also,

We get the linear system

$$\begin{array}{rcccccc} v & - & w & & & + & z & = & -100 \\ & & w & + & x & & & = & 500 \\ & & & & x & + & y & = & 400 \\ & & & & & & y & - & z & = & 100 \\ v & & & & & & & = & 100 \end{array}$$

and we solve by row reduction.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & -100 \\ 0 & 1 & 1 & 0 & 0 & 500 \\ 0 & 0 & 1 & 1 & 0 & 400 \\ 0 & 0 & 0 & 1 & -1 & 100 \\ 1 & 0 & 0 & 0 & 0 & 100 \end{array} \right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

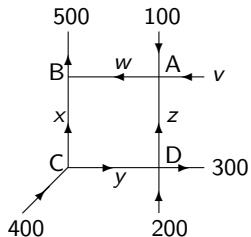
Then we go back to equation form...

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & | & 100 \\ 0 & 1 & 0 & 0 & -1 & | & 200 \\ 0 & 0 & 1 & 0 & 1 & | & 300 \\ 0 & 0 & 0 & 1 & -1 & | & 100 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{array}{l} v = 100 \\ w - z = 200 \\ x + z = 300 \\ y - z = 100 \end{array} \Rightarrow \begin{array}{l} v = 100 \\ w = 200 + z \\ x = 300 - z \\ y = 100 + z \end{array}$$

So

Note:

Solution:



$$v = 100$$

$$w = 200 + z$$

$$x = 300 - z$$

$$y = 100 + z$$

$$0 \leq z \leq 300$$

Additional questions:

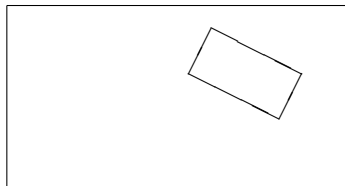
- 1 What are the minimum and maximum flows in the branch w ?

Answer:

- 2 If the flow along z is restricted to a maximum of 150 due to road construction, what is the maximum flow along y ?

Answer:

Weekend problem (for fun)

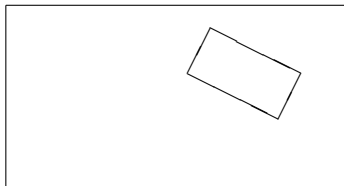


- You bake a delicious rectangular cake.
- In the middle of the night, your roommate gets hungry, sneaks into the kitchen, and cuts a rectangular piece out of your cake in a random position, at a random angle.

Question: How can you cut the cake in two, with a single straight cut, in such a way that each piece has the exact same amount of cake? You have only a knife and a straightedge (a ruler with no markings).

Note: There is one “cheating” answer and one “real” answer.

Weekend problem – Solution



Cheating answer:

Real answer:

Next time

For next time: Read Sections HSE, LC

- Today we developed a new language (vector and matrix equations) for discussing problems equivalent to linear systems
- Up to now, our final descriptions of solution set have been in the language of linear systems
- Next time we will discuss how to describe solution sets in terms of vectors – this will give us new insight into the solution sets