MAT 1302B – Mathematical Methods II

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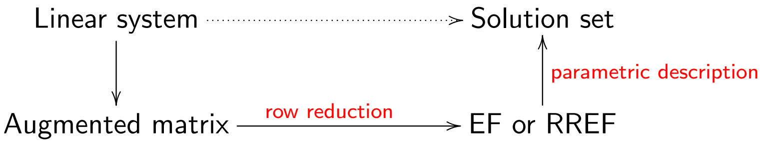
Winter 2015 – Lecture 4

Review

We have developed an algorithm for solving linear systems:



Important: Our algorithm is precise and involves no guesswork!

Even more important: This algorithm is one of the key concepts in this course. It is therefore absolutely essential that you practice solving problems until it is second nature to you!

Today

Develop a new language for discussing similar questions.

New language involves vectors and matrices.

New language will sometimes allow us to be more efficient and answer more complex questions.

Vectors

Definition

- A (column) vector is a matrix with only one column.
- $\mathbb{R}^n$ is the set of all vectors with n entries (i.e. $n \times 1$ matrices).

Examples

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ \frac{-2}{5} \end{bmatrix} \in \mathbb{R}^3, \qquad \vec{v} = \begin{bmatrix} 0 \\ -8 \end{bmatrix} \in \mathbb{R}^2,$$

$$\vec{w} = \begin{bmatrix} -7 \end{bmatrix} \in \mathbb{R}^1, \qquad \vec{z} = \begin{bmatrix} \frac{1}{2} \\ 0.53 \\ -2 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

Notes:

- $\in$ means "is an element of".
- Vectors are often written in bold (e.g. $\mathbf{u}$ instead of $\vec{u}$).

Vectors

Definition

Two vectors are equal iff (if and only if) they have the same number of entries and corresponding entries are equal.

Example:

$$\begin{bmatrix} 2 \\ -8 \end{bmatrix} \neq \begin{bmatrix} -8 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Vector addition

We add two vectors in $\mathbb{R}^n$ by adding corresponding entries.

Example:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ \frac{5}{6} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ not defined}$$

Vectors

Scalar multiplication

We multiply a vector $\vec{v}$ in $\mathbb{R}^n$ by a scalar (real number) c by multiplying each entry in $\vec{v}$ by c.

Example:

$$c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}, \quad (-2) \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \\ 8 \end{bmatrix}$$

Example

If

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

then

$$3\vec{u} + (-2)\vec{v} = 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} + \begin{bmatrix} -6 \\ -16 \end{bmatrix} = \begin{bmatrix} -3 \\ -22 \end{bmatrix}.$$

Notation

We sometimes write

$$(1,2) \quad \text{for} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad (-1,0,4) \quad \text{for} \quad \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \quad \text{etc.}$$

to save space.

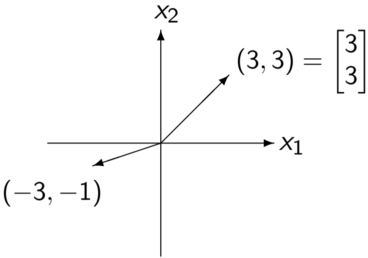
Note that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1,2) \neq \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \end{bmatrix}$ is not even a vector.

$\mathbb{R}^2$ – The plane

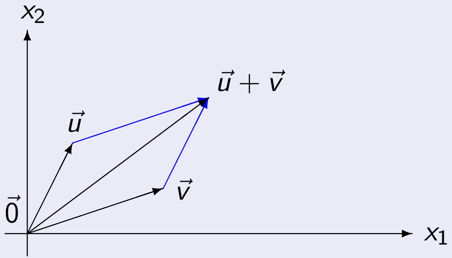
- Each point in the plane is determined by its cartesian coordinates – an ordered pair of real numbers.
- We write $P(a, b)$ for a point P with cartesian coordinates (a, b).
- We can identify the point $P(a, b)$ with the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and view $\mathbb{R}^2$ as the set of points in the plane.



Geometric interpretation of vector addition I

Parallelogram Rule

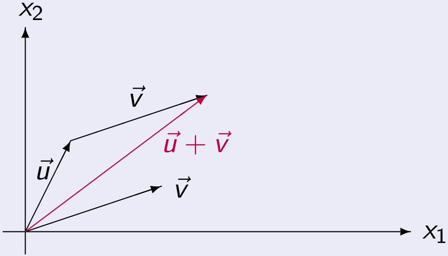
If $\vec{u}, \vec{v} \in \mathbb{R}^2$ are represented as points in the plane, then $\vec{u} + \vec{v}$ corresponds to the fourth vertex of a parallelogram whose other vertices are $\vec{u}$, $\vec{0}$ and $\vec{v}$.



Geometric interpretation of vector addition II

"Tip-to-tail"

Another geometric interpretation of adding vectors, is that we place them "tip-to-tail".



- We "slide" the vector $\vec{v}$ until its tail is at the tip of the vector $\vec{u}$.
- Then $\vec{u} + \vec{v}$ is the vector from the origin to the tip of $\vec{v}$.
- Note that this is really the same as the parallelogram rule.

Geometric interpretation of vector addition: Demonstration

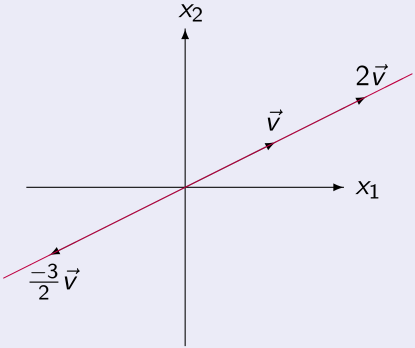
Wolfram Demonstrations Project

http://demonstrations.wolfram.com/2DVectorAddition/

Geometric interpretation of scalar multiples

Scalar multiples

Multiples of a given vector lie on a line



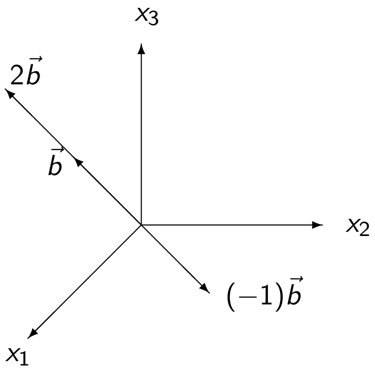
So

$$\{t\vec{v} \mid t \in \mathbb{R}\}$$

is the line through the origin in the direction of $\vec{v}$.

$\mathbb{R}^3$ – 3-dimensional space

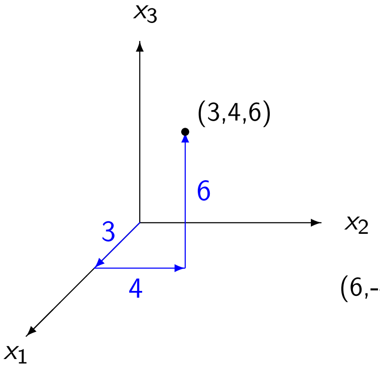
Vectors in $\mathbb{R}^3$ are represented by points in 3-dimensional space.

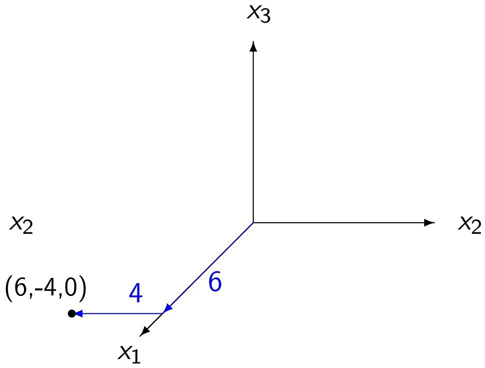


How to plot points in $\mathbb{R}^3$

To plot a point in $\mathbb{R}^3$, move in the direction of each axis, the number of units given by the coordinates of the point.

Example: Plot the points $(3, 4, 6)$ and $(6, -4, 0)$.

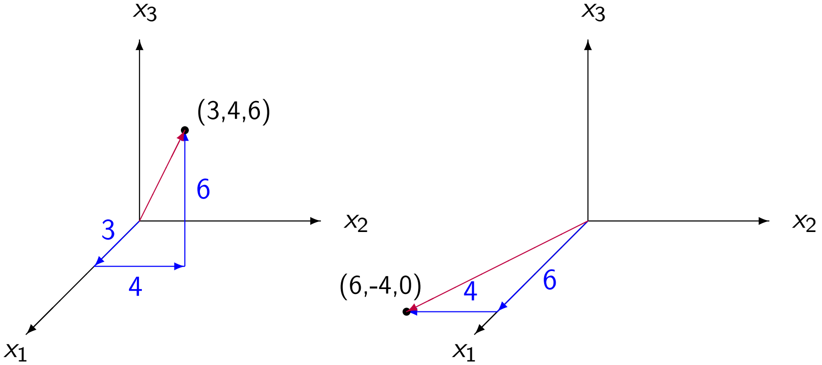




Drawing vectors in $\mathbb{R}^3$

To draw a vector in $\mathbb{R}^3$, plot the point with the same coordinates and then draw an arrow from the origin to that point.

Example: Plot the vectors $(3, 4, 6) \left(= \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right)$ and $(6, -4, 0) = \left(\begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix} \right)$.



Algebraic properties of $\mathbb{R}^n$

For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and scalars c and d, the following are true:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ — Commutativity
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ — Associativity
3. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ — $\vec{0}$ is an additive identity
4. $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$ — Note: $-\vec{u} = (-1)\vec{u}$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ — Distributivity
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ — Distributivity
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$ — 1 is a multiplicative identity

Subtraction of vectors

Question: How do we define $\vec{u} - \vec{v}$?

Answer: $\vec{u} - \vec{v} \stackrel{\text{def}}{=} \vec{u} + (-\vec{v})$.

Linear combinations

Definition

For $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_k$, the vector

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \tag{1}$$

is called a linear combination of $\vec{v}_1, \dots, \vec{v}_k$ with weights (or coefficients) $c_1, \dots, c_k$.

Example

$$\vec{v}_1 + \pi\vec{v}_2, \quad \frac{2}{3}\vec{v}_2 \quad (= 0\vec{v}_1 + \frac{2}{3}\vec{v}_2), \quad \vec{0} \quad (= 0\vec{v}_1 + 0\vec{v}_2)$$

are linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

Note: We think of the vector $\vec{u}$ in (1) as being obtained by traveling c_1 times the length of $\vec{v}_1$ in the direction of $\vec{v}_1$, then c_2 times the length of $\vec{v}_2$ in the direction of $\vec{v}_2$, etc.

Example

Question: Suppose

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}.$$

Can $\vec{b}$ be written as a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$? If so, find the weights/coefficients in such a linear combination.

Answer: We want to find scalars x_1, x_2, x_3 such that

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b},$$

$$\text{i.e.} \qquad x_1 \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}.$$

The equation

$$x_1 \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}$$

is the same as

$$\begin{bmatrix} 2x_1 \\ 2x_1 \\ -3x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 7x_3 \\ x_3 \\ 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}, \text{ hence } \begin{bmatrix} 2x_1 + 3x_2 + 7x_3 \\ 2x_1 + x_2 + x_3 \\ -3x_1 + x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}.$$

This is true iff

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 7x_3 & = & 0 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \\ -3x_1 & + & x_2 & + & 5x_3 & = & -11 \end{array}$$

This is a linear system!

We solve the linear system as before, using our algorithm

1. Write down the augmented matrix:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 7x_3 & = & 0 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \\ -3x_1 & + & x_2 & + & 5x_3 & = & -11 \end{array} \longrightarrow \left[\begin{array}{rrr|r} 2 & 3 & 7 & 0 \\ 2 & 1 & 1 & 4 \\ -3 & 1 & 5 & -11 \end{array}\right]$$

2. Then row reduce:

$$\left[\begin{array}{rrr|r} 2 & 3 & 7 & 0 \\ 2 & 1 & 1 & 4 \\ -3 & 1 & 5 & -11 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

3. No pivot in rightmost column $\implies$ there is a solution.
   No free variables $\implies$ the solution is unique.
   The only solution is

$$x_1 = 3, \quad x_2 = -2, \quad x_3 = 0.$$

Returning to our original question, the answer is YES. The vector $\vec{b}$ can be written as

$$\vec{b} = 3\vec{a}_1 - 2\vec{a}_2 + 0\vec{a}_3$$

and this is the only linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ that is equal to $\vec{b}$.

Check the answer!

$$3\vec{a}_1 - 2\vec{a}_2 + 0\vec{a}_3 = 3\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 - 2\cdot 3 + 0\cdot 7 \\ 3\cdot 2 - 2\cdot 1 + 0\cdot 1 \\ 3(-3) - 2\cdot 1 + 0\cdot 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix} = \vec{b}$$

Important note

The matrix we row reduced to see if $\vec{b}$ can be written as a linear combination of $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ was

$$\left[\begin{array}{ccc|c} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{b} \end{array}\right].$$

Vector equations and linear systems

Theorem (How to translate)

A vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_k\vec{a}_k = \vec{b}$$

has the same solution set as the LS with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k & \vec{b} \end{array}\right]. \tag{2}$$

In particular, $\vec{b}$ can be written as a linear combination of $\vec{a}_1, \dots, \vec{a}_k$ if and only if there exists a solution to the LS corresponding to (2).

So questions about linear combinations are closely related to questions about linear systems.

Therefore, we introduce a special name for the set of all linear combinations of a set of vectors – Span.

Span

Definition

If $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, then the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_k$ is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \ldots, \vec{v}_k$. It is denoted

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} \quad \text{or} \quad \langle \vec{v}_1, \ldots, \vec{v}_k \rangle.$$

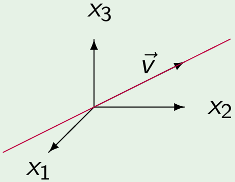
So $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ is the set of all vectors that can be written in the form

$$c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k$$

for some scalars $c_1, \ldots, c_k$.

Example 1 (Span of a single vector)

For $\vec{v} \in \mathbb{R}^3$, $\vec{v} \neq \vec{0}$, $\text{Span}\{\vec{v}\}$ is the set of all scalar multiples of $\vec{v}$.

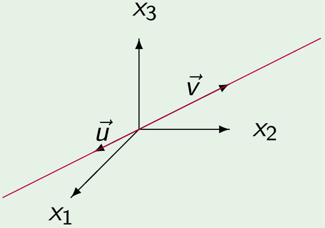


$\text{Span}\{\vec{v}\}$ is the line through the origin, in the direction given by $\vec{v}$.

Example 2 (Span of two vectors)

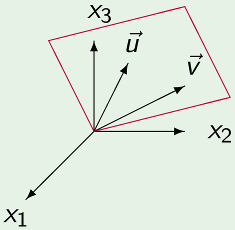
If $\vec{u}, \vec{v} \in \mathbb{R}^3$, $\vec{u}, \vec{v} \neq \vec{0}$, then

1. if $\vec{u}$ is a scalar multiple of $\vec{v}$,



Span$\{\vec{u}, \vec{v}\}$ is the line through the origin, parallel to $\vec{u}$ and $\vec{v}$.

2. if $\vec{u}$ is not a scalar multiple of $\vec{v}$,



Span$\{\vec{u}, \vec{v}\}$ is a plane through the origin containing $\vec{u}$ and $\vec{v}$.

Different languages

We have two languages for discussing what is essentially the same question.

Example 1

Write a system of equations that is equivalent to the vector equation

$$x_1 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 7 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -4 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 8 \end{bmatrix}.$$

Solution:

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & & & = & 3 \\ & & 2x_2 & + & 2x_3 & = & 4 \\ -x_1 & + & 7x_2 & - & 4x_3 & = & 0 \\ 4x_1 & + & 9x_2 & + & 10x_3 & = & 8 \end{array}$$

Different languages

Example 2

Write a vector equation that is equivalent to the following linear system:

$$\begin{array}{rcrcrcrcr} 6x_1 & + & 2x_2 & & & - & 4x_4 & = & 10 \\ & & -x_2 & + & 3x_3 & + & 3x_4 & = & -3 \end{array}$$

Solution:

$$x_1 \begin{bmatrix} 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$$

Different languages

Example 3

Suppose

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix}.$$

The following questions are equivalent:

1. Is $\vec{b}$ a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$?
2. Does the linear system

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 8 \\ 3x_1 & + & x_2 & + & x_3 & = & 1 \\ -x_1 & - & x_2 & + & 8x_3 & = & -2 \end{array}$$

have a solution?

Another example

Suppose

$$\vec{u} = (1, -1, 2), \quad \vec{v} = (0, 1, 0), \quad \vec{b} = (2, -3, 5).$$

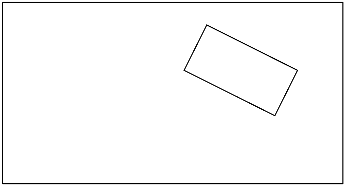
Is $\vec{u}$ a scalar multiple of $\vec{v}$? NO. (No multiple of the first coordinate of $\vec{v}$ equals the first coordinate of $\vec{u}$.)

So Span$\{\vec{u}, \vec{v}\}$ is a plane. Is $\vec{b}$ in this plane?

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & -3 \\ 2 & 0 & 5 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] \quad \text{No solution}$$

So the answer is NO, $\vec{b}$ is not in Span$\{\vec{u}, \vec{v}\}$.

Weekend problem (for fun)



- You bake a delicious rectangular cake.
- In the middle of the night, your roommate gets hungry, sneaks into the kitchen, and cuts a rectangular piece out of your cake in a random position, at a random angle.

Question: How can you cut the cake in two, with a single straight cut, in such a way that each piece has the exact same amount of cake? You have only a knife and a straightedge (a ruler with no markings).

Note: There is one "cheating" answer and one "real" answer.

Next time

For next time: Read Section RREF.MVNSE.

Next time we will:

- We have seen how problems about linear systems are equivalent to "geometric" problems involving vectors.
- Next time we will further develop this new language of vector and matrix equations.
- Application: Traffic/network flow. We will see how certain questions about the flow in a network amounts to solving linear systems.