MAT 1302B – Mathematical Methods II

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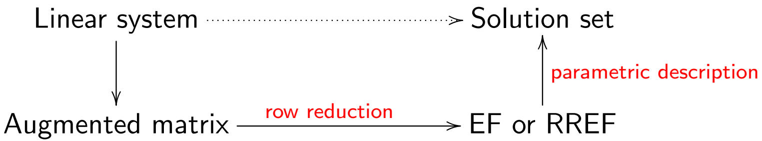
These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Review

We have developed an algorithm for solving linear systems:



Important: Our algorithm is precise and involves no guesswork!

Even more important: This algorithm is one of the key concepts in this course. It is therefore absolutely essential that you practice solving problems until it is second nature to you!

Today

Develop a new language for discussing similar questions.

New language involves vectors and matrices.

New language will sometimes allow us to be more efficient and answer more complex questions.

Vectors

Definition

- A (column) vector is a matrix with only one column.
- $\mathbb{R}^n$ is the set of all vectors with n entries (i.e. $n \times 1$ matrices).

Examples

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ \frac{-2}{5} \end{bmatrix} \in \_\_, \qquad \vec{v} = \begin{bmatrix} 0 \\ -8 \end{bmatrix} \in \_\_,$$

$$\vec{w} = \begin{bmatrix} -7 \end{bmatrix} \in \_\_, \qquad \vec{z} = \begin{bmatrix} \frac{1}{2} \\ 0.53 \\ -2 \\ 0 \end{bmatrix} \in \_\_$$

Notes:

- $\in$ means "is an element of".
- Vectors are often written in bold (e.g. $\mathbf{u}$ instead of $\vec{u}$).

Vectors

Definition

Two vectors are equal iff (if and only if) they have the same number of entries and corresponding entries are equal.

Example:

$$\begin{bmatrix} 2 \\ -8 \end{bmatrix} \quad \begin{bmatrix} -8 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Vector addition

We add two vectors in $\mathbb{R}^n$ by adding corresponding entries.

Example:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \qquad , \qquad \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ \frac{1}{3} \end{bmatrix} = \qquad , \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Vectors

Scalar multiplication

We multiply a vector $\vec{v}$ in $\mathbb{R}^n$ by a scalar (real number) c by multiplying each entry in $\vec{v}$ by c.

Example:

$$c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \qquad , \quad (-2) \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix} =$$

Example

If

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

then

$$3\vec{u} + (-2)\vec{v} =$$

Notation

We sometimes write

$$(1, 2) \quad \text{for} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad (-1, 0, 4) \quad \text{for} \quad \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \quad \text{etc.}$$

to save space.

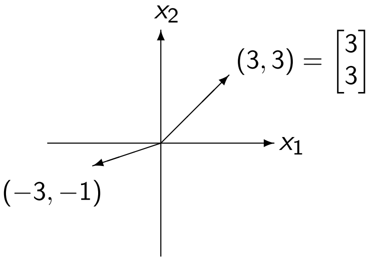
Note that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1, 2) \neq \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \end{bmatrix}$ is not even a vector.

$\mathbb{R}^2$ – The plane

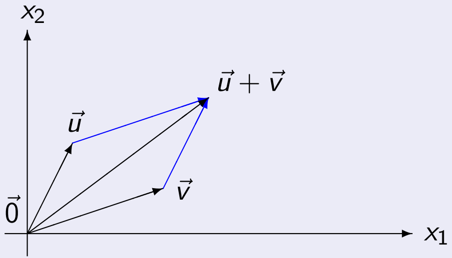
- Each point in the plane is determined by its cartesian coordinates – an ordered pair of real numbers.
- We write $P(a, b)$ for a point P with cartesian coordinates (a, b).
- We can identify the point $P(a, b)$ with the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and view $\mathbb{R}^2$ as the set of points in the plane.



Geometric interpretation of vector addition I

Parallelogram Rule

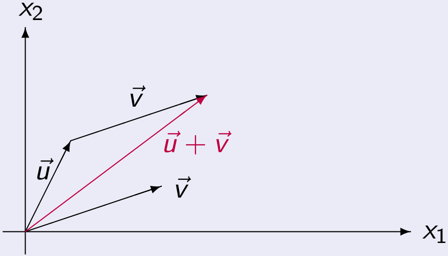
If $\vec{u}, \vec{v} \in \mathbb{R}^2$ are represented as points in the plane, then $\vec{u} + \vec{v}$ corresponds to the fourth vertex of a parallelogram whose other vertices are $\vec{u}$, $\vec{0}$ and $\vec{v}$.



Geometric interpretation of vector addition II

"Tip-to-tail"

Another geometric interpretation of adding vectors, is that we place them "tip-to-tail".



- We "slide" the vector $\vec{v}$ until its tail is at the tip of the vector $\vec{u}$.
- Then $\vec{u} + \vec{v}$ is the vector from the origin to the tip of $\vec{v}$.
- Note that this is really the same as the parallelogram rule.

Geometric interpretation of vector addition: Demonstration

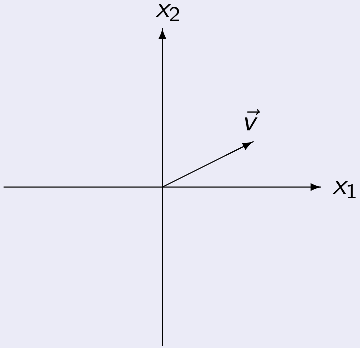
Wolfram Demonstrations Project

http://demonstrations.wolfram.com/2DVectorAddition/

Geometric interpretation of scalar multiples

Scalar multiples

Multiples of a given vector lie on a line

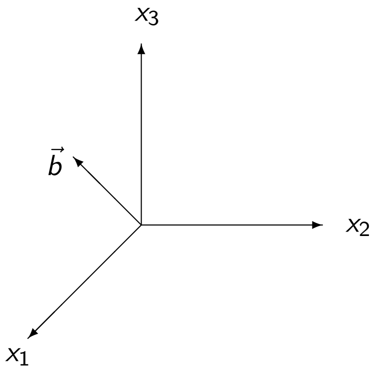


So

$$\{t\vec{v} \mid t \in \mathbb{R}\}$$

$\mathbb{R}^3$ – 3-dimensional space

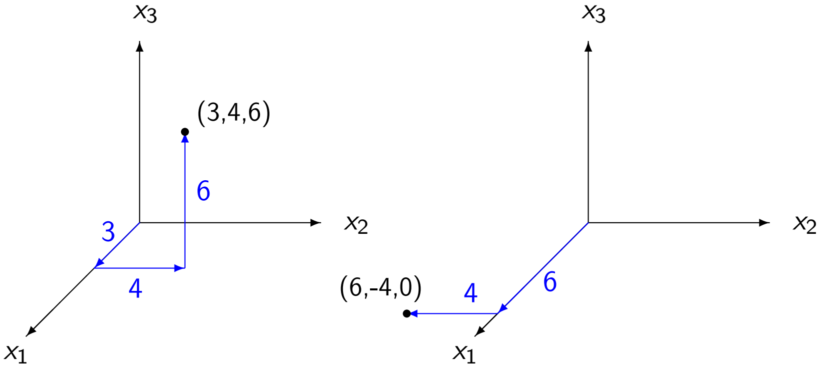
Vectors in $\mathbb{R}^3$ are represented by points in 3-dimensional space.



How to plot points in $\mathbb{R}^3$

To plot a point in $\mathbb{R}^3$, move in the direction of each axis, the number of units given by the coordinates of the point.

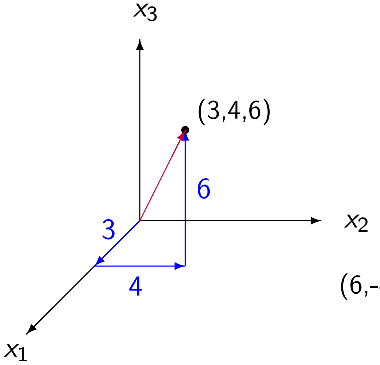
Example: Plot the points $(3, 4, 6)$ and $(6, -4, 0)$.

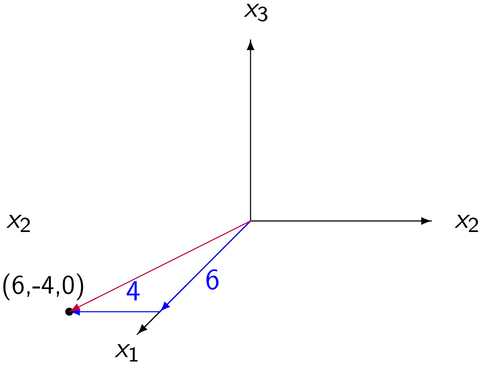


Drawing vectors in $\mathbb{R}^3$

To draw a vector in $\mathbb{R}^3$, plot the point with the same coordinates and then draw an arrow from the origin to that point.

Example: Plot the vectors $(3,4,6) \left(= \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right)$ and $(6,-4,0) = \left(\begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix} \right)$.





Algebraic properties of $\mathbb{R}^n$

For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and scalars c and d, the following are true:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$

Note: $-\vec{u} = (-1)\vec{u}$

Subtraction of vectors

Question: How do we define $\vec{u} - \vec{v}$?

Answer:

Linear combinations

Definition

For $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_k$, the vector

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \tag{1}$$

is called a linear combination of $\vec{v}_1, \dots, \vec{v}_k$ with weights (or coefficients) $c_1, \dots, c_k$.

Example

$$\vec{v}_1 + \pi\vec{v}_2, \quad \frac{2}{3}\vec{v}_2 \quad \left(= 0\vec{v}_1 + \frac{2}{3}\vec{v}_2\right), \quad \vec{0} \quad (= 0\vec{v}_1 + 0\vec{v}_2)$$

are linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

Note: We think of the vector $\vec{u}$ in (1) as being obtained by traveling c_1 times the length of $\vec{v}_1$ in the direction of $\vec{v}_1$, then c_2 times the length of $\vec{v}_2$ in the direction of $\vec{v}_2$, etc.

Example

Question: Suppose

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}.$$

Can $\vec{b}$ be written as a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$? If so, find the weights/coefficients in such a linear combination.

Answer: We want to find scalars x_1, x_2, x_3 such that

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b},$$

$$\text{i.e.} \qquad x_1 \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}.$$

The equation

$$x_1 \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}$$

is the same as

$$\begin{bmatrix} 2x_1 \\ 2x_1 \\ -3x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 7x_3 \\ x_3 \\ 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}, \text{ hence } \begin{bmatrix} 2x_1 + 3x_2 + 7x_3 \\ 2x_1 + x_2 + x_3 \\ -3x_1 + x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -11 \end{bmatrix}.$$

This is true iff

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 7x_3 & = & 0 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \\ -3x_1 & + & x_2 & + & 5x_3 & = & -11 \end{array}$$

This is a linear system!

We solve the linear system as before, using our algorithm

1. Write down the augmented matrix:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 7x_3 & = & 0 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \\ -3x_1 & + & x_2 & + & 5x_3 & = & -11 \end{array} \longrightarrow \left[\begin{array}{rrr|r} 2 & 3 & 7 & 0 \\ 2 & 1 & 1 & 4 \\ -3 & 1 & 5 & -11 \end{array}\right]$$

2. Then row reduce:

$$\left[\begin{array}{rrr|r} 2 & 3 & 7 & 0 \\ 2 & 1 & 1 & 4 \\ -3 & 1 & 5 & -11 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

3.

Returning to our original question, the answer is ____.

Check the answer!

Important note

The matrix we row reduced to see if $\vec{b}$ can be written as a linear combination of $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ was

$$\left[\begin{array}{ccc|c} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{b} \end{array}\right].$$

Vector equations and linear systems

Theorem (How to translate)

A vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_k\vec{a}_k = \vec{b}$$

has the same solution set as the LS with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k & \vec{b} \end{array}\right]. \tag{2}$$

In particular, $\vec{b}$ can be written as a linear combination of $\vec{a}_1, \ldots, \vec{a}_k$ if and only if there exists a solution to the LS corresponding to (2).

So questions about linear combinations are closely related to questions about linear systems.

Therefore, we introduce a special name for the set of all linear combinations of a set of vectors – Span.

Span

Definition

If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_k$ is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \dots, \vec{v}_k$. It is denoted

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \quad \text{or} \quad \langle \vec{v}_1, \dots, \vec{v}_k \rangle.$$

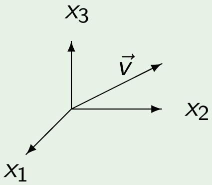
So $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is the set of all vectors that can be written in the form

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

for some scalars $c_1, \dots, c_k$.

Example 1 (Span of a single vector)

For $\vec{v} \in \mathbb{R}^3$, $\vec{v} \neq \vec{0}$, $\operatorname{Span}\{\vec{v}\}$ is ____________________________

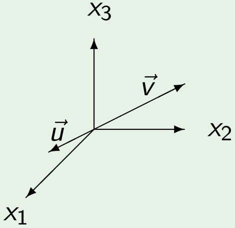


$\operatorname{Span}\{\vec{v}\}$ is

Example 2 (Span of two vectors)

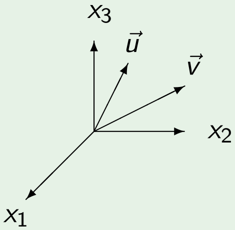
If $\vec{u}, \vec{v} \in \mathbb{R}^3$, $\vec{u}, \vec{v} \neq \vec{0}$, then

1. if $\vec{u}$ is a scalar multiple of $\vec{v}$,



Span$\{\vec{u}, \vec{v}\}$ is

2. if $\vec{u}$ is not a scalar multiple of $\vec{v}$,



Span$\{\vec{u}, \vec{v}\}$ is

Different languages

We have two languages for discussing what is essentially the same question.

Example 1

Write a system of equations that is equivalent to the vector equation

$$x_1 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 7 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -4 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 8 \end{bmatrix}.$$

Solution:

Different languages

Example 2

Write a vector equation that is equivalent to the following linear system:

$$\begin{array}{rcrcrcrcr} 6x_1 & + & 2x_2 & & & - & 4x_4 & = & 10 \\ & & -x_2 & + & 3x_3 & + & 3x_4 & = & -3 \end{array}$$

Solution:

Different languages

Example 3

Suppose

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix}.$$

The following questions are equivalent:

1. Is $\vec{b}$ a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$?
2. Does the linear system

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 8 \\ 3x_1 & + & x_2 & + & x_3 & = & 1 \\ -x_1 & - & x_2 & + & 8x_3 & = & -2 \end{array}$$

have a solution?

Another example

Suppose

$$\vec{u} = (1, -1, 2), \quad \vec{v} = (0, 1, 0), \quad \vec{b} = (2, -3, 5).$$

Is $\vec{u}$ a scalar multiple of $\vec{v}$? ____

So Span$\{\vec{u}, \vec{v}\}$ is _______

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & -3 \\ 2 & 0 & 5 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$$

So the answer is

Weekend problem (for fun)

Next time

For next time: Read Section RREF.MVNSE.

Next time we will:

- We have seen how problems about linear systems are equivalent to "geometric" problems involving vectors.
- Next time we will further develop this new language of vector and matrix equations.
- Application: Traffic/network flow. We will see how certain questions about the flow in a network amounts to solving linear systems.