MAT 1302B – Mathematical Methods II

Alistair Savage

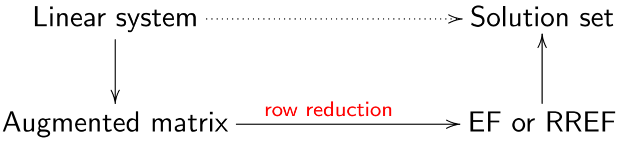
Mathematics and Statistics
University of Ottawa

Winter 2015 – Lecture 3

Review

Goal: Develop an algorithm for solving LS's.

Technique:



Last time: We developed an algorithm for row reduction.

Today: We focus on the right vertical arrow.

Question

Suppose we have reduced the augmented matrix of a LS to EF or RREF. What is the solution set?

Example

Example

Suppose we have reduced the augmented matrix of a LS to the matrix

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & -4 & 0 & 3 \\ 0 & 0 & 1 & 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right] \qquad \text{RREF}$$

What is the solution set?

We return to equation form:

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & & & - & 4x_5 & & & = & 3 \\ & & & & x_3 & & & + & 5x_5 & & & = & 2 \\ & & & & & & x_4 & - & x_5 & & & = & -8 \\ & & & & & & & & & & x_6 & = & 6 \end{array}$$

$$\begin{array}{ccccccccccc} x_1 & + & 2x_2 & & & & - & 4x_5 & & = & 3 \\ & & & & x_3 & & + & 5x_5 & & = & 2 \\ & & & & & x_4 & - & x_5 & & = & -8 \\ & & & & & & & & x_6 & = & 6 \end{array}$$

- The variables corresponding to pivot columns are called basic variables or leading variables.
- The other variables are called free variables.

We solve the reduced system for the basic variables in terms of the free variables to get the general solution:

$$\begin{aligned} x_1 &= 3 - 2x_2 + 4x_5 \\ x_3 &= 2 - 5x_5 \\ x_4 &= -8 + x_5 \\ x_6 &= 6 \\ x_2, x_5 &\ \text{free} \end{aligned}$$

(basic variables on the left, constants and free variables on the right).

$$\begin{aligned} x_1 &= 3 - 2x_2 + 4x_5 \\ x_3 &= 2 - 5x_5 \\ x_4 &= -8 + x_5 \\ x_6 &= 6 \\ x_2, x_5 &\text{ free} \end{aligned}$$

- We can choose any values we like for the free variables and the above equations determine the values of the basic variables.
- Each choice of values for the free variables gives a solution.
- The solution set consists of all of these solutions.

Example

$$x_2 = 1, \quad x_5 = 0$$

$$\implies x_1 = 1, \quad x_3 = 2, \quad x_4 = -8, \quad x_6 = 6$$

Thus $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 2, -8, 0, 6)$ is one solution. But it's only one of many.

A description of the solution set (as in previous example)

$$\begin{aligned} x_1 &= 3 - 2x_2 + 4x_5 \\ x_3 &= 2 - 5x_5 \\ x_4 &= -8 + x_5 \\ x_6 &= 6 \\ x_2, x_5 &\text{ free} \end{aligned}$$

is called a parametric description and the free variables act as parameters.

Set notation

The solution set is

$$\{(3 - 2x_2 + 4x_5, x_2, 2 - 5x_5, -8 + x_5, x_5, 6) \mid x_2, x_5 \in \mathbb{R}\}.$$

Note

A solution set can have many parametric descriptions. Our algorithm gives one of these – the standard form.

Remark

Sometimes we introduce new letters to denote the parameters.

One thinks of these as the numbers that are substituted for the free variables.

So the previous general solution would be written as

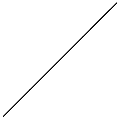
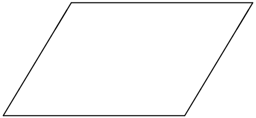
$$\begin{aligned} x_1 &= 3 - 2s + 4t \\ x_3 &= 2 - 5t \\ x_4 &= -8 + t \\ x_6 &= 6 \\ s, t &\in \mathbb{R} \end{aligned}$$

and the solution set would be written

$$\{(3 - 2s + 4t, s, 2 - 5t, -8 + t, t, 6) \mid s, t \in \mathbb{R}\}.$$

Geometric interpretation

The number of parameters (free variables) in a parametric description of a solution set has a geometric interpretation.

parameters	geometric interpretation of solution set	
0	point	
1	line	
2	plane	

Note: We are assuming here the system is consistent.

Geometric interpretation: Example

Consider the linear system in three variables x, y, z:

$$\begin{array}{rcrcl} & & -3z & = & 0 \\ 7x & - & 4z & = & 0 \\ 7x & & & = & 0 \end{array}$$

We write the augmented matrix and row reduce

$$\left[\begin{array}{ccc|c} 0 & 0 & -3 & 0 \\ 7 & 0 & -4 & 0 \\ 7 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 7 & 0 & -4 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{-R_1 + R_2} \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{-3}{4} R_2 + R_3} \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{-1}{4} R_2} \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7} R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Geometric interpretation: Example

We have row reduced the augmented matrix to RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Returning to equation notation gives

$$\begin{aligned} x \qquad &= 0 \\ z &= 0 \end{aligned}$$

The basic variables are x and z. The free variable is y. So the general solution is

$$x = 0$$
$$y \quad \text{free}$$
$$z = 0$$

and the solution set is

$$\{(0, y, 0) \mid y \in \mathbb{R}\}. \qquad \longleftarrow \text{a line}$$

Geometric interpretation: Example

Recall, our general solution is

$$x = 0$$
$$y \quad \text{free}$$
$$z = 0$$

- Solutions to each individual equation in original system correspond to a plane.
- Fact that solution set has one parameter (free variable) means that solution set forms a line.
- So the three planes corresponding to the three equations intersect in a line.

Demonstration of this solution set

http://demonstrations.wolfram.com/PlanesSolutionsAnd
GaussianEliminationOfA33LinearSystem/

Examples

Suppose we have row reduced the augmented matrix of a LS to the following matrices.

1. Are the systems consistent?
2. If so, how many parameters are in the description of the solution set and what is the geometric interpretation of this solution set?

Example 1

$$\left[\begin{array}{cccc|c} 1 & -5 & 3 & 10 & 7 \\ 0 & 1 & 2 & 8 & 8 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

- Inconsistent $\implies$ no solutions (even though there is 1 free variable).

Example 2

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Consistent (no contradictions).
- 1 parameter.
- Solution set is a line.

Example 3

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & -3 & 0 & 0 & 10 \\ 0 & 0 & 1 & 2 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

- Consistent (no contradictions).
- 2 parameters.
- Solution set is a plane.

Questions

Suppose a LS consists of a equations in b variables

Question 1

What is the maximum number of leading 1's in the RREF of the coefficient matrix (= number of pivot positions in the coefficient matrix)?

Answer:

$$\text{minimum of } a \text{ and } b$$

since the coefficient matrix has a rows and b columns and each pivot position is in its own row and column.

Question 2

What is the maximum number of pivots positions in the augmented matrix?

Answer:

$$\text{minimum of } a \text{ and } b+1$$

Question 3

If the coefficient matrix has n pivot positions, how many can the augmented matrix have?

Answer: n or $n+1$ since every pivot position in the coefficient matrix is also a pivot position in the augmented matrix and the augmented matrix has one extra column.

Example:

$$\left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{array}\right] \quad \text{vs.} \quad \left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & & 1 & \\ & & & & 1 \end{array}\right]$$

Example

Suppose a LS consists of 6 equations in 4 unknowns.

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

# pivot pos in CM	# pivot pos in AM	possible?	consistent?	# params in sol. set
5	5	N	-	-
4	5	Y	N	-
4	4	Y	Y	0 (point)
3	4	Y	N	-
3	3	Y	Y	1 (line)
2	2	Y	Y	2 (plane)

Existence and Uniqueness

Existence and Uniqueness Theorem

A LS is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form

$$0 \cdots\cdots 0 \mid b, \qquad b \neq 0.$$

If a LS is consistent, then

1. the solution set contains a unique solution when there are no free variables,
2. the solution set contains infinitely many solutions when is one or more free variables.

Note: This justifies a statement we made earlier that a linear system has zero, one or infinitely many solutions.

Solving linear systems

We now have a precise algorithm for solving any LS:

1. Write the augmented matrix.
2. Use row reduction to reduce the augmented matrix to echelon form (remember we developed a precise algorithm for doing this).
3. If there is no solution (i.e. there is a row of the form

$$0 \cdots \cdots 0 \mid b, \qquad b \neq 0 \qquad)$$

   stop. Otherwise, continue.
4. Continue row reduction to obtain RREF.
5. Write the system of equations corresponding to the RREF.
6. Solve for the basic variables in terms of the free variables (if there are any). This gives the general solution in standard form.
7. If desired, write the solution set in set notation.

Example

Example

Consider the LS

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & -1 \\ 4x_1 & + & ax_2 & = & b \end{array}$$

For what values of a and b does the system have

1. no solution,
2. exactly one solution, or
3. infinitely many solutions?

Answer: First we try to row reduce the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 4 & a & b \end{array}\right] \xrightarrow{-4R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & a-8 & b+4 \end{array}\right]$$

Example (cont.)

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 4 & a & b \end{array}\right] \xrightarrow{-4R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & a-8 & b+4 \end{array}\right]$$

1. The system has no solution when

$$a - 8 = 0 \quad \text{and} \quad b + 4 \neq 0$$

   i.e when $a = 8$ and $b \neq -4$.

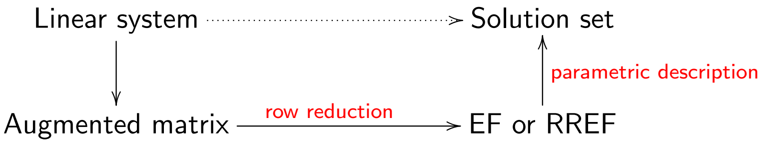
2. The system has exactly one solution when $a - 8 \neq 0$, i.e. $a \neq 8$.
3. The system has infinitely many solutions when

$$a - 8 = 0 \quad \text{and} \quad b + 4 = 0$$

   i.e. $a = 8$ and $b = -4$.

Solving linear systems

We now have an algorithm for solving linear systems:



Important: Our algorithm is precise and involves no guesswork! It's just a series of steps to follow (compare to differentiation versus integration).

Even more important: This algorithm is one of the key concepts in this course. It is therefore absolutely essential that you practice solving problems until it is second nature to you!

One more example: start to finish

Solve the following linear system:

$$\begin{aligned} 2x_2 + x_1 + 2x_4 + 3x_5 &= 14 + 4x_3 + x_1 \\ 2x_1 + 3x_3 - x_4 + 4x_5 &= -7 + 2x_1 + x_3 + x_2 \\ 8x_3 + 28 &= 4x_2 + 4x_4 + 4x_5 \end{aligned}$$

1. Put the system in standard form and write the augmented matrix.

$$\begin{array}{rcrcrcrcl} 2x_2 & - & 4x_3 & + & 2x_4 & + & 3x_5 & = & 14 \\ -x_2 & + & 2x_3 & - & x_4 & + & 4x_5 & = & -7 \\ -4x_2 & + & 8x_3 & - & 4x_4 & - & 4x_5 & = & -28 \end{array}$$

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & -1 & 2 & -1 & 4 & -7 \\ 0 & -4 & 8 & -4 & -4 & -28 \end{array}\right]$$

2. Row reduce to EF.

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & -1 & 2 & -1 & 4 & -7 \\ 0 & -4 & 8 & -4 & -4 & -28 \end{array}\right] \xrightarrow[2R_1+R_3]{\frac{1}{2}R_1+R_2} \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & \frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{4}{11}R_2+R_3} \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & \frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{EF}$$

3. The system is consistent since the rightmost column is not a pivot column (no row of the form $0 \cdots 0 \mid b, \quad b \neq 0$). So we continue.
4. Continue row reduction to RREF.

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & \frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{2}{11}R_2} \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 0 & 14 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccccc|c} 0 & 1 & -2 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{RREF}$$

5. Write the system of equations corresponding to the RREF.

$$\begin{array}{ccccccccc} x_2 & - & 2x_3 & + & x_4 & & & = & 7 \\ & & & & & & x_5 & = & 0 \end{array}$$

6. Solve for the basic variables in terms of the free variables.

The basic variables are x_2 and x_5. The free variables are x_1, x_3, and x_4. So the general solution is

$$x_2 = 2x_3 - x_4 + 7$$

$$x_5 = 0$$

$$x_1, x_3, x_4 \quad \text{free}$$

Solution set: $\{(x_1, 2x_3 - x_4 + 7, x_3, x_4, 0) \mid x_1, x_3, x_4 \in \mathbb{R}\}$

Check your answer!

Original system:

$$\begin{aligned} 2x_2 + x_1 + 2x_4 + 3x_5 &= 14 + 4x_3 + x_1 \\ 2x_1 + 3x_3 - x_4 + 4x_5 &= -7 + 2x_1 + x_3 + x_2 \\ 8x_3 + 28 &= 4x_2 + 4x_4 + 4x_5 \end{aligned}$$

General solution:

$$\begin{aligned} x_2 &= 2x_3 - x_4 + 7 \\ x_5 &= 0 \\ x_1, x_3, x_4 &\quad \text{free} \end{aligned}$$

Substitute in original system for basic variables in terms of free variables:

$$\begin{aligned} 2(2x_3 - x_4 + 7) + x_1 + 2x_4 + 3 \cdot 0 &= 14 + 4x_3 + x_1 \quad \checkmark \\ 2x_1 + 3x_3 - x_4 + 4 \cdot 0 &= -7 + 2x_1 + x_3 + (2x_3 - x_4 + 7) \quad \checkmark \\ 8x_3 + 28 &= 4(2x_3 - x_4 + 7) + 4x_4 + 4 \cdot 0 \quad \checkmark \end{aligned}$$

Weekend problem (from last time)

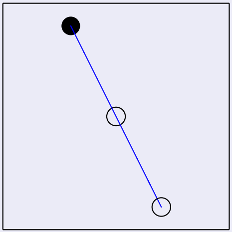
"Quarter game" rules:

- Square table.
- You and your opponent take turns placing quarters on the table.
- Quarter must be placed in an empty space.
- Game ends when one player can't play – that player loses.

Problem: If you are allowed the first move, how can you play to guarantee you will win?

Weekend problem – Solution

Strategy



- Place your first quarter in the centre.
- Every time your opponent places a quarter, place your quarter in the position the same distance from the centre, but in the opposite direction.

Why will you win?

- After each of your turns, the board is symmetric in the centre (i.e. each point of the table is occupied if and only if the point on the opposite side of the centre, the same distance away, is occupied).
- Thus, if your opponent can play another quarter in some position, the opposite position is always free for you to place your quarter!

Next time

For next time:

- Do recommended exercises
- Read Sections VO, LC, SS

Next time we will:

- Develop a new language for discussing similar questions
- New language involves vectors and matrices
- New language will sometimes allow us to be more efficient and answer more complex questions