MAT 1302B – Mathematical Methods II

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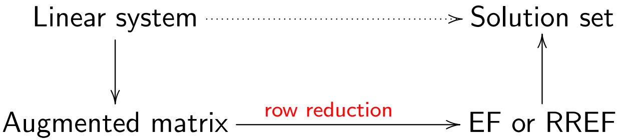
Winter 2015 – Lecture 3

These are partial slides for following along in class. Full versions of these slides will be posted on the course website after the lecture.

Review

Goal: Develop an algorithm for solving LS's.

Technique:



Last time: We developed an algorithm for row reduction.

Today: We focus on the right vertical arrow.

Question

Suppose we have reduced the augmented matrix of a LS to EF or RREF. What is the solution set?

Example

Example

Suppose we have reduced the augmented matrix of a LS to the matrix

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & -4 & 0 & 3 \\ 0 & 0 & 1 & 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right] \qquad \rule{2cm}{0.4pt}$$

What is the solution set?

We return to equation form:

$$\begin{array}{rcccccl} x_1 + 2x_2 & & & - & 4x_5 & & = 3 \\ & x_3 & & + & 5x_5 & & = 2 \\ & & x_4 & - & x_5 & & = -8 \\ & & & & & x_6 & = 6 \end{array}$$

$$
\begin{array}{ccccccccccccc}
x_1 & + & 2x_2 & & & & & - & 4x_5 & & & = & 3 \\
& & & & x_3 & & & + & 5x_5 & & & = & 2 \\
& & & & & & x_4 & - & x_5 & & & = & -8 \\
& & & & & & & & & & x_6 & = & 6
\end{array}
$$

- The variables corresponding to pivot columns are called ______________________________
- The other variables are called ____________

We solve the reduced system for the basic variables in terms of the free variables to get the general solution:

$$
\begin{aligned}
x_1 &= 3 - 2x_2 + 4x_5 \\
x_3 &= 2 - 5x_5 \\
x_4 &= -8 + x_5 \\
x_6 &= 6 \\
x_2, x_5 &\text{ free}
\end{aligned}
$$

$$\begin{aligned} x_1 &= 3 - 2x_2 + 4x_5 \\ x_3 &= 2 - 5x_5 \\ x_4 &= -8 + x_5 \\ x_6 &= 6 \\ x_2, x_5 &\text{ free} \end{aligned}$$

- We can choose any values we like for the ____________ and the above equations determine the values of the ______________
- Each choice of values for the ____________ gives a solution.
- The solution set consists of all of these solutions.

Example

$$x_2 = 1, \quad x_5 = 0$$

$$\implies \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Thus $(x_1, x_2, x_3, x_4, x_5, x_6) =$ _______________ is one solution. But it's only one of many.

A description of the solution set (as in previous example)

$$\begin{aligned} x_1 &= 3 - 2x_2 + 4x_5 \\ x_3 &= 2 - 5x_5 \\ x_4 &= -8 + x_5 \\ x_6 &= 6 \\ x_2, x_5 &\text{ free} \end{aligned}$$

is called a parametric description and the free variables act as parameters.

Set notation

The solution set is

$$\{(3 - 2x_2 + 4x_5, x_2, 2 - 5x_5, -8 + x_5, x_5, 6) \mid x_2, x_5 \in \mathbb{R}\}.$$

Note

A solution set can have many parametric descriptions. Our algorithm gives one of these – the standard form.

Remark

Sometimes we introduce new letters to denote the parameters.

One thinks of these as the numbers that are substituted for the free variables.

So the previous general solution would be written as

$$\begin{aligned} x_1 &= 3 - 2s + 4t \\ x_3 &= 2 - 5t \\ x_4 &= -8 + t \\ x_6 &= 6 \\ s, t &\in \mathbb{R} \end{aligned}$$

and the solution set would be written

$$\{(3 - 2s + 4t, s, 2 - 5t, -8 + t, t, 6) \mid s, t \in \mathbb{R}\}.$$

Geometric interpretation

The number of parameters (free variables) in a parametric description of a solution set has a geometric interpretation.

parameters	geometric interpretation of solution set
0	____
1	___
2	____

Note: We are assuming here the system is consistent.

Geometric interpretation: Example

Consider the linear system in three variables x, y, z:

$$\begin{array}{rcrcl} & & -3z & = & 0 \\ 7x & - & 4z & = & 0 \\ 7x & & & = & 0 \end{array}$$

We write the augmented matrix and row reduce

$$\left[\begin{array}{ccc|c} 0 & 0 & -3 & 0 \\ 7 & 0 & -4 & 0 \\ 7 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 7 & 0 & -4 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Geometric interpretation: Example

We have row reduced the augmented matrix to RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Returning to equation notation gives

$$\begin{aligned} x \quad\quad &= 0 \\ z &= 0 \end{aligned}$$

The basic variables are _______ The free variable is _. So the general solution is

and the solution set is

Geometric interpretation: Example

Recall, our general solution is

- Solutions to each individual equation in original system correspond to a plane.
- Fact that solution set has one parameter (free variable) means that solution set forms a line.
- So the three planes corresponding to the three equations intersect in a line.

Demonstration of this solution set

http://demonstrations.wolfram.com/PlanesSolutionsAndGaussianEliminationOfA33LinearSystem/

Examples

Suppose we have row reduced the augmented matrix of a LS to the following matrices.

1. Are the systems consistent?
2. If so, how many parameters are in the description of the solution set and what is the geometric interpretation of this solution set?

Example 1

$$\left[\begin{array}{cccc|c} 1 & -5 & 3 & 10 & 7 \\ 0 & 1 & 2 & 8 & 8 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

- ____

Example 2

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- ____________________________
- ___________
- ___________________

Example 3

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & -3 & 0 & 0 & 10 \\ 0 & 0 & 1 & 2 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

- ____________________________
- ____________
- ____________________

Questions

Suppose a LS consists of a equations in b variables

Question 1

What is the maximum number of leading 1's in the RREF of the coefficient matrix (= number of pivot positions in the coefficient matrix)?

Answer:

Question 2

What is the maximum number of pivots positions in the augmented matrix?

Answer:

Question 3

If the coefficient matrix has n pivot positions, how many can the augmented matrix have?

Answer:

Example:

Example

Suppose a LS consists of 6 equations in 4 unknowns.

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

# pivot pos in CM	# pivot pos in AM	possible?	consistent?	# params in sol. set
5	5			
4	5			
4	4			
3	4			
3	3			
2	2			

Existence and Uniqueness

Existence and Uniqueness Theorem

A LS is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form

$$0 \cdots\cdots 0 \mid b, \qquad b \neq 0.$$

If a LS is consistent, then

1. the solution set contains a unique solution when there are no free variables,
2. the solution set contains infinitely many solutions when is one or more free variables.

Note: This justifies a statement we made earlier that a linear system has zero, one or infinitely many solutions.

Solving linear systems

We now have a precise algorithm for solving any LS:

1. Write the augmented matrix.
2. Use row reduction to reduce the augmented matrix to echelon form (remember we developed a precise algorithm for doing this).
3. If there is no solution (i.e. there is a row of the form

   $$0 \cdots\cdots 0 \mid b, \qquad b \neq 0 \qquad)$$

   stop. Otherwise, continue.
4. Continue row reduction to obtain RREF.
5. Write the system of equations corresponding to the RREF.
6. Solve for the basic variables in terms of the free variables (if there are any). This gives the general solution in standard form.
7. If desired, write the solution set in set notation.

Example

Example

Consider the LS

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & -1 \\ 4x_1 & + & ax_2 & = & b \end{array}$$

For what values of a and b does the system have

1. no solution,
2. exactly one solution, or
3. infinitely many solutions?

Answer: First we try to row reduce the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 4 & a & b \end{array}\right]$$

Example (cont.)

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 4 & a & b \end{array}\right]$$

1. The system has no solution when

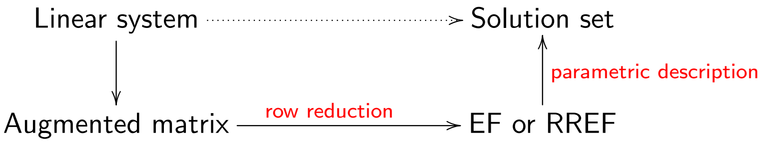
   ____

2. The system has exactly one solution when ________
3. The system has infinitely many solutions when

   ____

Solving linear systems

We now have an algorithm for solving linear systems:



Important: Our algorithm is precise and involves no guesswork! It's just a series of steps to follow (compare to differentiation versus integration).

Even more important: This algorithm is one of the key concepts in this course. It is therefore absolutely essential that you practice solving problems until it is second nature to you!

One more example: start to finish

Solve the following linear system:

$$\begin{aligned} 2x_2 + x_1 + 2x_4 + 3x_5 &= 14 + 4x_3 + x_1 \\ 2x_1 + 3x_3 - x_4 + 4x_5 &= -7 + 2x_1 + x_3 + x_2 \\ 8x_3 + 28 &= 4x_2 + 4x_4 + 4x_5 \end{aligned}$$

1. Put the system in standard form and write the augmented matrix.

$$\begin{array}{rcrcrcrcl} 2x_2 & - & 4x_3 & + & 2x_4 & + & 3x_5 & = & 14 \\ -x_2 & + & 2x_3 & - & x_4 & + & 4x_5 & = & -7 \\ -4x_2 & + & 8x_3 & - & 4x_4 & - & 4x_5 & = & -28 \end{array}$$

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & -1 & 2 & -1 & 4 & -7 \\ 0 & -4 & 8 & -4 & -4 & -28 \end{array}\right]$$

2. Row reduce to EF.

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & -1 & 2 & -1 & 4 & -7 \\ 0 & -4 & 8 & -4 & -4 & -28 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & \frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & \frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

3. The system is

4. Continue row reduction to RREF.

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & \frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 3 & 14 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 0 & 2 & -4 & 2 & 0 & 14 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 0 & 1 & -2 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{RREF}$$

5. Write the system of equations corresponding to the RREF.

$$\begin{aligned} x_2 - 2x_3 + x_4 \quad &= 7 \\ x_5 &= 0 \end{aligned}$$

6. Solve for the basic variables in terms of the free variables.

The basic variables are ________. The free variables are ____________. So the general solution is

Solution set:

Check your answer!

Original system:

$$\begin{aligned} 2x_2 + x_1 + 2x_4 + 3x_5 &= 14 + 4x_3 + x_1 \\ 2x_1 + 3x_3 - x_4 + 4x_5 &= -7 + 2x_1 + x_3 + x_2 \\ 8x_3 + 28 &= 4x_2 + 4x_4 + 4x_5 \end{aligned}$$

General solution:

Substitute in original system for basic variables in terms of free variables:

Weekend problem (from last time)

"Quarter game" rules:

- Square table.
- You and your opponent take turns placing quarters on the table.
- Quarter must be placed in an empty space.
- Game ends when one player can't play – that player loses.

Problem: If you are allowed the first move, how can you play to guarantee you will win?

Weekend problem – Solution

Next time

For next time:

- Do recommended exercises
- Read Sections VO, LC, SS

Next time we will:

- Develop a new language for discussing similar questions
- New language involves vectors and matrices
- New language will sometimes allow us to be more efficient and answer more complex questions