MAT 1302B – Mathematical Methods II

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Overview

Announcements

DGDs start next week.

Last Time

- Overview of linear algebra
- Linear equations
- Systems of linear equations (linear systems)
- Solution sets
- Matrices (coefficient matrix and augmented matrix)

Today's goal

Develop a precise procedure for solving any system.

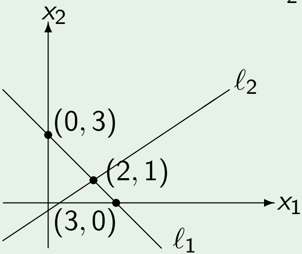
Geometric interpretation of linear systems: 2 variables

Recall

- We have considered some linear systems in two variables.
- If we plot the solutions to each equation individually, we get lines.
- Solutions to the system correspond to points on the intersection of the lines.

Example (from last class)

$$\begin{array}{rcrcrcl} \ell_1 & : & x_1 & + & x_2 & = & 3 \\ \ell_2 & : & 2x_1 & - & 3x_2 & = & 1 \end{array}$$



- The pair of numbers (s_1, s_2) satisfies both equations if it lies on both lines.
- The solution set consists of the single solution $(2, 1)$.

Geometric interpretation of linear systems: 3 variables

Solutions of a single equation

Question: If we plot the solutions to a single equation in three variables, what do we get?

Answer: A plane!

Example

- Consider the linear system

$$\begin{array}{rcrcrcl} 4x & + & 2y & + & 3z & = & 3 \\ x & + & 2y & + & 3z & = & 3 \\ -x & - & 2y & - & z & = & 3 \end{array}$$

- The solutions to each individual equation form a plane.
- The solutions to the system correspond to the points of intersection of the planes.

Geometric interpretation of linear systems: 3 variables

Question

- What type of solution sets are possible?
- How can planes intersect?

Wolfram Mathematica Player

Download from

http://www.wolfram.com/cdf-player/

Demonstration of solving a 3×3 linear system

http://demonstrations.wolfram.com/PlanesSolutionsAnd
GaussianEliminationOfA33LinearSystem/

(all one word)

Matrix terminology

Matrix terminology

- zero row or column: row or column with all entries equal to zero
- nonzero row or column: row or column with at least one nonzero entry
- leading entry of a row: leftmost nonzero entry

Examples

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 8 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

Terminology

(Row) Echelon form

A matrix is in (row) echelon form if it satisfies the following 3 conditions:

1. All nonzero rows are above all zero rows.
2. The leading entry of a nonzero row is to the right of the leading entry of any row above it.
3. All entries in a column below a leading entry are zeros (this actually follows from the second condition).

Reduced (row) echelon form

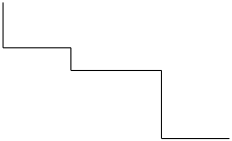
A matrix is in reduced (row) echelon form if it satisfies the above 3 properties and:

4. All leading entries are 1s.
5. Each leading 1 is in the only nonzero entry in its column.

Terminology

A matrix in echelon (respectively reduced echelon) form is called an echelon matrix (respectively reduced echelon matrix).

Note: The word "echelon" means *step-like formation* (military, etc.).



Examples

Example 1

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Reduced row echelon form (RREF)

Example 2

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 0 & \frac{-1}{2} & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row echelon form (EF)

Examples

Example 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Not in echelon form

Example 4

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & \frac{-1}{2} & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Not in echelon form

Examples (cont.)

Example 5

$$\begin{bmatrix} 0 & 0 & \boxed{\neq 0} & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{\neq 0} & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\neq 0} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\neq 0} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{EF}$$

($*$=arbitrary entry)

Example 6

$$\begin{bmatrix} 0 & 0 & 1 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{RREF}$$

Reduced row echelon form

Theorem

Every matrix is row equivalent to exactly one reduced echelon matrix.

Important note: A matrix may be row equivalent to many echelon matrices.

Example

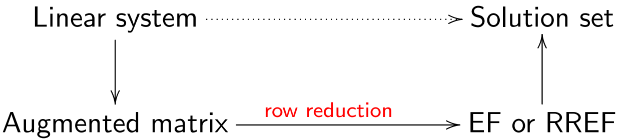
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

are all row equivalent and are all in echelon form. However, only the leftmost matrix is in reduced row echelon form.

Overview

Goal: Develop an algorithm for solving LS's.

Technique:



We first focus on the bottom arrow (row reduction). We will forget about LS's for the time being.

Let's work through another example and try to keep track of why we are performing each step.

Example

Reduce the following matrix to EF.

$$\begin{bmatrix} 0 & 0 & -1 & 2 & -7 \\ 2 & 1 & 3 & 2 & 5 \\ 2 & 1 & 3 & 6 & 2 \\ -4 & -2 & -7 & -2 & -17 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & -1 & 2 & -7 \\ 2 & 1 & 3 & 6 & 2 \\ -4 & -2 & -7 & -2 & -17 \end{bmatrix}$$

$$\xrightarrow[2R_1+R_4]{-R_1+R_3} \begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & -1 & 2 & -7 \\ 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 2 & -7 \end{bmatrix} \xrightarrow{-R_2+R_4} \begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & -1 & 2 & -7 \\ 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{EF}$$

Pivot positions and columns

Definition

- A pivot position in a matrix A is a location that corresponds to a leading 1 in the RREF of A (or a leading entry in an EF of A).
- A pivot column is a column of A that contains a pivot position.

Previous example

$$\begin{bmatrix} \boxed{0} & 0 & -1 & 2 & -7 \\ 2 & 1 & \boxed{3} & 2 & 5 \\ 2 & 1 & 3 & \boxed{6} & 2 \\ -4 & -2 & -7 & -2 & -17 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} \boxed{2} & 1 & 3 & 2 & 5 \\ 0 & 0 & \boxed{-1} & 2 & -7 \\ 0 & 0 & 0 & \boxed{4} & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ EF}$$

The boxed positions are the pivot positions and the first, third and fourth columns are the pivot columns.

Pivots

Definition

A pivot is a nonzero number in a pivot position used to create zeros via row operations.

Previous example

$$\begin{bmatrix} 0 & 0 & -1 & 2 & -7 \\ 2 & 1 & 3 & 2 & 5 \\ 2 & 1 & 3 & 6 & 2 \\ -4 & -2 & -7 & -2 & -17 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & -1 & 2 & -7 \\ 2 & 1 & 3 & 6 & 2 \\ -4 & -2 & -7 & -2 & -17 \end{bmatrix}$$

$$\xrightarrow[2R_1+R_4]{-R_1+R_3} \begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & -1 & 2 & -7 \\ 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 2 & -7 \end{bmatrix} \xrightarrow{-R_2+R_4} \begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & -1 & 2 & -7 \\ 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{EF}$$

The pivots are 2 and -1 (4 would also be a pivot if we continue to RREF).

Row reduction algorithm (Gauss-Jordan elimination)

Example: Reduce the following to EF and then RREF

$$\begin{bmatrix} 0 & \boxed{0} & 2 & 4 & 2 & 0 & 10 \\ 0 & -1 & -2 & -3 & -1 & 0 & -3 \\ 0 & 6 & 6 & 6 & 9 & 0 & 6 \\ 0 & 3 & -1 & -5 & -1 & 0 & -20 \end{bmatrix}$$

Step 1: Begin with the leftmost nonzero column. This is a pivot column and the pivot position is at the top.

Step 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & \boxed{-1} & -2 & -3 & -1 & 0 & -3 \\ 0 & 0 & 2 & 4 & 2 & 0 & 10 \\ 0 & 6 & 6 & 6 & 9 & 0 & 6 \\ 0 & 3 & -1 & -5 & -1 & 0 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \boxed{-1} & -2 & -3 & -1 & 0 & -3 \\ 0 & 0 & 2 & 4 & 2 & 0 & 10 \\ 0 & 6 & 6 & 6 & 9 & 0 & 6 \\ 0 & 3 & -1 & -5 & -1 & 0 & -20 \end{bmatrix}$$

Step 3: Use row replacement operations to create zeros in all positions below the pivot (i.e. add appropriate multiples of row containing the pivot to the rows below it).

$$\xrightarrow[3R_1+R_4]{6R_1+R_3} \begin{bmatrix} 0 & -1 & -2 & -3 & -1 & 0 & -3 \\ 0 & 0 & 2 & 4 & 2 & 0 & 10 \\ 0 & 0 & -6 & -12 & 3 & 0 & -12 \\ 0 & 0 & -7 & -14 & -4 & 0 & -29 \end{bmatrix}$$

Step 4: Ignore (cover) the row containing the pivot position and all rows above it. Apply steps 1-3 to the remaining submatrix. Repeat this process until there are no more nonzero rows to modify.

$$
\left[\begin{array}{ccccccc}
0 & -1 & -2 & -3 & -1 & 0 & -3 \\ \hline
0 & 0 & \boxed{2} & 4 & 2 & 0 & 10 \\
0 & 0 & -6 & -12 & 3 & 0 & -12 \\
0 & 0 & -7 & -14 & -4 & 0 & -29
\end{array}\right]
$$

$$
\xrightarrow[]{\substack{3R_2+R_3 \\ \frac{7}{2}R_2+R_4}}
\left[\begin{array}{ccccccc}
0 & -1 & -2 & -3 & -1 & 0 & -3 \\ \hline
0 & 0 & \boxed{2} & 4 & 2 & 0 & 10 \\
0 & 0 & 0 & 0 & 9 & 0 & 18 \\
0 & 0 & 0 & 0 & 3 & 0 & 6
\end{array}\right]
$$

$$
\left[\begin{array}{ccccccc}
0 & -1 & -2 & -3 & -1 & 0 & -3 \\
0 & 0 & 2 & 4 & 2 & 0 & 10 \\ \hline
0 & 0 & 0 & 0 & \boxed{9} & 0 & 18 \\
0 & 0 & 0 & 0 & 3 & 0 & 6
\end{array}\right]
$$

$$
\xrightarrow{-\frac{1}{3}R_3+R_4}
\left[\begin{array}{ccccccc}
0 & -1 & -2 & -3 & -1 & 0 & -3 \\
0 & 0 & 2 & 4 & 2 & 0 & 10 \\ \hline
0 & 0 & 0 & 0 & \boxed{9} & 0 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\quad \text{EF}
$$

Step 5: Beginning with the rightmost pivot and working up/left, use a scaling operation to make each pivot a 1 and use replacement to create zeros above each pivot.

$$\begin{bmatrix} 0 & -1 & -2 & -3 & -1 & 0 & -3 \\ 0 & 0 & 2 & 4 & 2 & 0 & 10 \\ 0 & 0 & 0 & 0 & \boxed{9} & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{9}R_3} \begin{bmatrix} 0 & -1 & -2 & -3 & -1 & 0 & -3 \\ 0 & 0 & 2 & 4 & 2 & 0 & 10 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[-2R_3+R_2]{R_3+R_1} \begin{bmatrix} 0 & -1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 0 & 2 & 4 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 0 & \boxed{2} & 4 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 0 & -1 & -2 & -3 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{2R_2+R_1} \begin{bmatrix} 0 & \boxed{-1} & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_1} \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

Another example

$$
\begin{bmatrix} \boxed{2} & 4 & 4 & 3 & 13 & -6 \\ -2 & -4 & -4 & 0 & -7 & 8 \\ 4 & 8 & 4 & 0 & 20 & -2 \end{bmatrix}
\xrightarrow[-2R_1+R_3]{R_1+R_2}
\begin{bmatrix} 2 & 4 & 4 & 3 & 13 & -6 \\ 0 & 0 & \boxed{0} & 3 & 6 & 2 \\ 0 & 0 & -4 & -6 & -6 & 10 \end{bmatrix}
$$

$$
\xrightarrow{R_2 \leftrightarrow R_3}
\begin{bmatrix} 2 & 4 & 4 & 3 & 13 & -6 \\ 0 & 0 & -4 & -6 & -6 & 10 \\ 0 & 0 & 0 & \boxed{3} & 6 & 2 \end{bmatrix}
\xrightarrow{\frac{1}{3}R_3}
\begin{bmatrix} 2 & 4 & 4 & 3 & 13 & -6 \\ 0 & 0 & -4 & -6 & -6 & 10 \\ 0 & 0 & 0 & \boxed{1} & 2 & \frac{2}{3} \end{bmatrix}
$$

$$
\xrightarrow[6R_3+R_2]{-3R_3+R_1}
\begin{bmatrix} 2 & 4 & 4 & 0 & 7 & -8 \\ 0 & 0 & \boxed{-4} & 0 & 6 & 14 \\ 0 & 0 & 0 & 1 & 2 & \frac{2}{3} \end{bmatrix}
\xrightarrow{\frac{-1}{4}R_2}
\begin{bmatrix} 2 & 4 & 4 & 0 & 7 & -8 \\ 0 & 0 & \boxed{1} & 0 & \frac{-3}{2} & \frac{-7}{2} \\ 0 & 0 & 0 & 1 & 2 & \frac{2}{3} \end{bmatrix}
$$

$$
\xrightarrow{-4R_2+R_1}
\begin{bmatrix} \boxed{2} & 4 & 0 & 0 & 13 & 6 \\ 0 & 0 & 1 & 0 & \frac{-3}{2} & \frac{-7}{2} \\ 0 & 0 & 0 & 1 & 2 & \frac{2}{3} \end{bmatrix}
\xrightarrow{\frac{1}{2}R_1}
\begin{bmatrix} 1 & 2 & 0 & 0 & \frac{13}{2} & 3 \\ 0 & 0 & 1 & 0 & \frac{-3}{2} & \frac{-7}{2} \\ 0 & 0 & 0 & 1 & 2 & \frac{2}{3} \end{bmatrix}
$$

Different algorithms

Our algorithm allows you to reduce any augmented matrix to EF or RREF.

The algorithm is not unique – there are other ones that work (e.g. algorithm described in the proof of Theorem REMEF in the FCLA text).

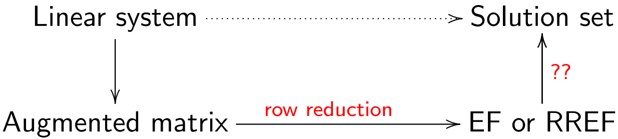
Since the RREF of any matrix is unique, any valid algorithm will give you the same RREF in the end (assuming you don't make any mistakes).

Reducing a matrix to EF or RREF is called row reduction.

- Reducing a matrix to EF is called Gaussian elimination.
- Reducing a matrix to RREF is called Gauss-Jordan elimination.

Solutions of linear systems

Recall



Suppose the augmented matrix of a LS has been row reduced to the following RREF's. What is the solution set of the LS?

Example 1

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right] \qquad \begin{array}{rcl} x_1 & = & -3 \\ x_2 & = & 2 \\ x_3 & = & 5 \end{array}$$

Only solution is

$$x_1 = -3, \quad x_2 = 2, \quad x_3 = 5.$$

Solutions of linear systems (cont.)

Example 2

$$\left[\begin{array}{ccccc|c} 0 & 1 & 6 & 11 & 18 & 0 \\ 0 & 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{array}\right] \quad \longleftrightarrow \quad 0 = 9 \quad \text{contradiction}$$

The solution set is empty (i.e. no solution).

Example 3

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right] \qquad \begin{array}{rcl} x_1 + 2x_3 & = & 0 \\ x_2 - x_3 & = & 2 \\ x_4 & = & 8 \end{array}$$

The variables corresponding to pivot columns (or leading 1's in RREF) are called basic variables (x_1, x_2, x_4 here) and the others are called free variables (x_3 here).

Solutions of linear systems (cont.)

Example 3 (cont.)

We have the reduced system:

$$\begin{aligned} x_1 \quad\quad + 2x_3 \quad &= 0 \\ x_2 - x_3 \quad &= 2 \\ x_4 &= 8 \end{aligned}$$

We solve the reduced system of equations for the basic variables in terms of the free variables.

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= 2 + x_3 \\ x_3 &\text{ free} \\ x_4 &= 8 \end{aligned}$$

Solutions of linear systems (cont.)

Example 3 (cont.)

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= 2 + x_3 \\ x_3 &\text{ free} \\ x_4 &= 8 \end{aligned} \tag{1}$$

Each basic variable occurs in exactly one equation. The free variables (x_3 here) can have any value and the above equations determine the values of the basic variables.

Example: If $x_3 = 2$, the corresponding solution is

$$(x_1, x_2, x_3, x_4) = (-4, 4, 2, 8).$$

(1) is called the general solution of the LS because it gives an explicit description of all solutions.

Check your answer!

Checking your answer when there are free variables

Question: We should always check our answer. How do we do that when there is more than one solution?

Can we check each one? NO! There are an infinite number of solutions!

Answer: We should replace each basic variable with its expression in terms of the free variables. If we didn't make any mistakes, all of the terms with free variables should cancel.

Check your answer!

Example 3 (cont.)

Our system was

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right] \qquad \begin{array}{ccccccccc} x_1 & & & + & 2x_3 & & & = & 0 \\ & & x_2 & - & x_3 & & & = & 2 \\ & & & & & & x_4 & = & 8 \end{array}$$

and our solution was

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= 2 + x_3 \\ x_3 &\ \text{free} \\ x_4 &= 8 \end{aligned}$$

Substituting gives

$$\begin{array}{cccccccc} (-2x_3) & & & + & 2x_3 & & = & 0 & \checkmark \\ & & (2+x_3) & - & x_3 & & = & 2 & \checkmark \\ & & & & & 8 & = & 8 & \checkmark \end{array}$$

Weekend problem (for fun)

"Quarter game" rules:

- Square table
- You and your opponent take turns placing quarters on the table
- Quarter must be placed in an empty space
- Game ends when one player can't play – that player loses

Problem: If you are allowed the first move, how can you play to guarantee you will win?

Next time

To Do:

- Read Section TSS of the text.
- Do the recommended exercises.

Next Time:

- More on general solutions
- Geometric interpretation of linear systems
- Existence of solutions (when is a system consistent?)
- Uniqueness of solutions (is there just one solution or many?)