MAT 1302B – Mathematical Methods II

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Winter 2015 – Lecture 1

Course webpage:
http://mysite.science.uottawa.ca/asavag2/mat1302

Overview

Webpage: `http://mysite.science.uottawa.ca/asavag2/mat1302`

Keys to success:

- After each class, do the recommended exercises for that class before the next lecture
- Read the section in the text before each lecture
- Attend class and the DGDs

If you have difficulties:

- As soon as you have a difficulty, resolve it!
  - come to office hours
  - go to the help centre
  - ask the TA during the DGD
- The course will become very difficult if you fall behind

Note: DGDs begin week of January 19.

Taking notes

Partial slides will be posted before each class. Full slides will be posted after each class. Download a copy to your personal computer or USB flash drive in case the server goes down before an exam!

Video of each lecture will be made available for streaming online.

You should not try to write down everything presented in class (you won't have enough time and you can download the slides).

Instead, you should focuss on writing down the key ideas, important concepts, potential pitfalls, etc.

If you don't understand something at first, but then it clears up after further examples done in class, you should write yourself a note about that issue to remind yourself later.

What is linear algebra?

Algebra

- Branch of mathematics studying structure, relations and quantity
- Symbolic operations, variables
- Systematic solutions of equations, algorithms

Linear

- Geometric: relating to lines
- Algebraic: directly proportional (e.g. $y = mx$, m a constant)

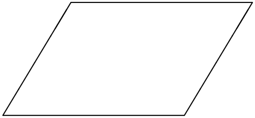
Linear algebra

Study of vectors, matrices, linear transformations and systems of linear equations

Linear geometry

Vectors and lines are linear

Planes are linear



(3-dimensional) space is linear

Linear equations

Question

What is a linear equation? Possible ideas:

- Should be the equation of a linear geometric object (i.e. equation of a line, plane, etc.).
  Complication: How do we know this just by looking at an equation? What about more variables (i.e. high dimensions we can't picture)?

- Purely algebraic definition – how can we make this precise?

Example

$$2x + 3y = 6$$

Example

$$2x + 3y + 0 \cdot z = 6$$

Note: We will often use $x_1, x_2, x_3, \ldots$ instead of x, y, z.

Linear equations: Definition

What do the above examples have in common?

All the variables appear to the first power.

Definition (Linear Equation)

A linear equation in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n, b \in \mathbb{R}$ are the coefficients and n can be any positive integer.

Linear equations: Examples

Example 1

$$-x_1 + 5x_2 - 3x_3 = 1$$

is a linear equation.

Example 2

$$2x_1 - 5 = 3x_2 + x_1 - 8$$

is a linear equation since it can be rewritten as

$$x_1 - 3x_2 = -3.$$

Example 3

$$\sqrt{2}(x_1 - x_2) = \pi(5 - x_1)$$

is a linear equation.

Linear equations: Examples

Example 4

$$(2x_1 - 1)(3 - x_2) = 0$$

is not a linear equation since it is equivalent to

$$6x_1 + x_2 - 2x_1x_2 = 3$$

and cannot be simplified to remove the nonlinear term $2x_1x_2$.

Example 5

$$x_1^2 + x_2 = 3$$

is not linear.

Linear equations: Examples

Example 6

$$\sqrt{x_1} + x_2 = 0$$

is not linear.

Example 7

$$x_1^2 - 1 = (x_1 + 1)(x_1 - 1) \qquad (1)$$

$$\begin{aligned} (1) &\iff x_1^2 - 1 = x_1^2 - 1 \\ &\iff 0 = 0 \end{aligned}$$

So (1) is a linear equation.

Recall: An equation is linear if it can be written in the form

$$a_1 x_1 + \cdots + a_n x_n = b.$$

Systems of linear equations

Definition (System of linear equations)

A system of linear equations (or linear system) is a collection of one or more linear equations involving the same variables.

Example

$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ & & 5x_2 & - & x_3 & = & -1 \end{array}$$

Definition (Solution of a linear system)

A solution of the system is a list $(s_1, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$ (in that order).

Systems of linear equations: Example

Example

Recall our example

$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ & & 5x_2 & - & x_3 & = & -1 \end{array}$$

$(1,0,1)$, i.e. $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, is a solution of this system since substitution gives

$$\begin{aligned} 3(1) - 2(0) + (1) = 4 \quad & \checkmark \\ 5(0) - (1) = -1 \quad & \checkmark \end{aligned}$$

$(1,1,3)$, i.e. $x_1 = 1$, $x_2 = 1$, $x_3 = 3$, is not a solution since substitution gives

$$\begin{aligned} 3(1) - 2(1) + (3) = 4 \quad & \checkmark \\ 5(1) - (3) = -1 \quad & \times \end{aligned}$$

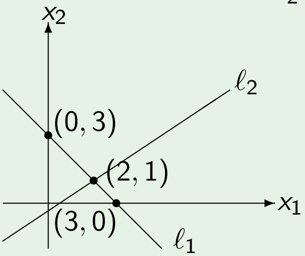
Solution sets

Definition (Solution set)

The solution set of a linear system is the set of all possible solutions of the system.

Example 1

$$\begin{array}{rcrcrcl} \ell_1 & : & x_1 & + & x_2 & = & 3 \\ \ell_2 & : & 2x_1 & - & 3x_2 & = & 1 \end{array}$$

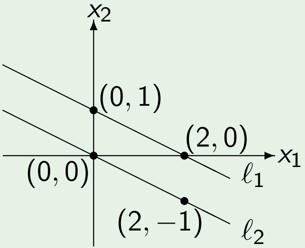


- The pair of numbers (s_1, s_2) satisfies both equations iff it lies on both lines.
- The solution set consists of the single solution $(2,1)$.

Solution sets

Example 2

$$\begin{array}{rcrcrcl} \ell_1 & : & x_1 & + & 2x_2 & = & 2 \\ \ell_2 & : & -x_1 & - & 2x_2 & = & 0 \end{array}$$

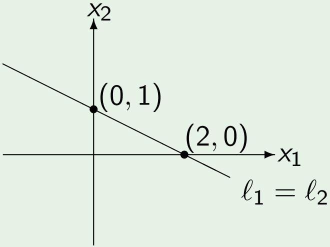


- Lines are parallel and have no points in common.
- The system has no solutions.
- The solution set is empty.

Solution sets

Example 3

$$\begin{aligned} \ell_1 &: x_1 + 2x_2 = 2 \\ \ell_2 &: 2x_1 + 4x_2 = 4 \end{aligned}$$



- Lines coincide (i.e. ℓ_1 and ℓ_2 are the same line).
- Any point on the line is a solution.
- There are infinitely many solutions!!

Possible solution sets

Question

What types of solutions sets can systems of linear equations have?

Answer

A system of linear equations has either

1. No solution — system is inconsistent
2. Exactly one solution
3. Infinitely many solutions

(2 and 3: system is consistent)

Note: This is special to linear systems.

Example: The equation $x^2 = 4$ has exactly 2 solutions ($x = \pm 2$) but it is not linear.

Matrix notation

In order to simplify calculations, we often encode the info of a linear system in a matrix.

Example

For the system

$$\begin{array}{rcrcrcrcr} 2x_1 & - & x_2 & - & x_3 & + & x_4 & = & 7 \\ & & x_2 & + & x_3 & + & 2x_4 & = & -8 \\ 5x_1 & & & - & 3x_3 & - & x_4 & = & 0 \end{array}$$

we have two matrices:

coefficient matrix

$$\begin{bmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 5 & 0 & -3 & -1 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{cccc|c} 2 & -1 & -1 & 1 & 7 \\ 0 & 1 & 1 & 2 & -8 \\ 5 & 0 & -3 & -1 & 0 \end{array}\right]$$

Matrix sizes

In our previous example:

coefficient matrix

$$\begin{bmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 5 & 0 & -3 & -1 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{cccc|c} 2 & -1 & -1 & 1 & 7 \\ 0 & 1 & 1 & 2 & -8 \\ 5 & 0 & -3 & -1 & 0 \end{array}\right]$$

The coefficient matrix has size 3×4 and the augmented matrix has size 3×5.

Definition

A matrix has size $m \times n$ if it has m rows and n columns.

Note: Sometimes (for instance, in the text) the vertical bar in the augmented matrix is omitted.

Question: What type of linear system is completely trivial to solve?

Answer: A system in which each equation involves only a single variable, with coefficient one.

Example

$$
\begin{aligned}
x_1 \qquad\qquad\qquad\qquad &= -1\\
x_2 \qquad\qquad\qquad &= 0\\
x_3 \qquad\qquad &= 5\\
x_4 &= \tfrac{1}{2}
\end{aligned}
$$

We can just read off the solution!

Moral

- Ideally, we would like to manipulate a system to turn it into an easy system like this.
- This won't always be possible, but we'll come close.

Example

Let's solve a particular linear system. In each step, we'll also write the augmented matrix.

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 2x_3 & = & -6 \\ & & x_2 & + & 3x_3 & = & 6 \\ 3x_1 & + & 4x_2 & - & x_3 & = & -5 \end{array} \qquad \left[\begin{array}{ccc|c} 2 & 4 & -2 & -6 \\ 0 & 1 & 3 & 6 \\ 3 & 4 & -1 & -5 \end{array}\right]$$

1. Multiply first equation by $1/2$.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 1x_3 & = & -3 \\ & & x_2 & + & 3x_3 & = & 6 \\ 3x_1 & + & 4x_2 & - & x_3 & = & -5 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & 3 & 6 \\ 3 & 4 & -1 & -5 \end{array}\right]$$

2. Add -3 times row 1 (R_1) to row 3 (R_3).

Example (cont.)

2. Add -3 times row 1 (R_1) to row 3 (R_3).

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 1x_3 & = & -3 \\ & & x_2 & + & 3x_3 & = & 6 \\ & & -2x_2 & + & 2x_3 & = & 4 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & 3 & 6 \\ 0 & -2 & 2 & 4 \end{array}\right]$$

3. Add 2 times R_2 to R_3.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 1x_3 & = & -3 \\ & & x_2 & + & 3x_3 & = & 6 \\ & & & & 8x_3 & = & 16 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 8 & 16 \end{array}\right]$$

The system is now in triangular form.

4. Multiply R_3 by $1/8$.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 1x_3 & = & -3 \\ & & x_2 & + & 3x_3 & = & 6 \\ & & & & x_3 & = & 2 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example (cont.)

5. (a) Add R_3 to R_1.
   (b) Add $(-3)R_3$ to R_2.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & -1 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 2 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

6. Add $(-2)R_2$ to R_1.

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -1 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 2 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: The unique solution is

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 2.$$

Important: Check your answer!

Question

How can we check that our solution is correct? Maybe we made some mistakes along the way!

Answer

Substitute our solution(s) into the original system of equations!

Example

In our example, we substitute $x_1 = -1$, $x_2 = 0$, $x_3 = 2$ into the system

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 2x_3 & = & -6 \\ & & x_2 & + & 3x_3 & = & 6 \\ 3x_1 & + & 4x_2 & - & x_3 & = & -5 \end{array} \quad \rightarrow \quad \begin{array}{rcrcrcr} 2(-1) & + & 4(0) & - & 2(2) & = & -6 \checkmark \\ & & (0) & + & 3(2) & = & 6 \checkmark \\ 3(-1) & + & 4(0) & - & 2 & = & -5 \checkmark \end{array}$$

Elementary row operations

When solving linear systems, we will use three basic row operations.

Elementary row operations

1. Replacement: Replace one row by the sum of itself and a multiple of another row (i.e. add a multiple of one row to another row).
2. Interchange: Interchange two rows.
3. Scale: Multiply all entries in a row by a nonzero constant.

Definition

Two matrices are row equivalent if one can be obtained from the other by a sequence of elementary row operations.

Row equivalence: Example

Example

We see from our previous example that the matrices

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & -6 \\ 0 & 1 & 3 & 6 \\ 3 & 4 & -1 & -5 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

are row equivalent since we obtained the second from the first via row operations.

Row operations: reversibility

Note: All the row operations are reversible.

Examples

1. Operation: Add cR_1 to R_2.
   Reverse: Add $-cR_1$ to R_2.
2. Operation: Interchange R_2 and R_3.
   Reverse: Interchange R_2 and R_3.
3. Operation: Multiply R_2 by c, $c \neq 0$.
   Reverse: Multiply R_2 by $1/c$.

Row equivalence and solution sets

- If a matrix A is obtained from a matrix B by row operations, then any solution of the system corresponding to A is a solution of the system corresponding to B.
- Since A can also be obtained from B by row operations, any solution of the system corresponding to B is a solution of the system corresponding to A.

Theorem

If the augmented matrices of two linear systems (LS's) are row equivalent, then the two systems have the same solution set.

In our example, we solved the LS by finding a row equivalent one which was simpler and allowed us to read off the solution.

Another Example

Determine if the following system is consistent:

$$\begin{array}{rcrcrcr} & & -2x_2 & + & x_3 & = & 2 \\ -x_1 & + & 2x_2 & + & 3x_3 & = & -1 \\ -2x_1 & + & 6x_2 & + & 5x_3 & = & -1 \end{array}$$

Solution: The augmented matrix is

$$\left[\begin{array}{rrr|r} 0 & -2 & 1 & 2 \\ -1 & 2 & 3 & -1 \\ -2 & 6 & 5 & -1 \end{array}\right]$$

We use row operations to get it to triangular form.

Another Example (cont.)

$$\left[\begin{array}{ccc|c} 0 & -2 & 1 & 2 \\ -1 & 2 & 3 & -1 \\ -2 & 6 & 5 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} -1 & 2 & 3 & -1 \\ 0 & -2 & 1 & 2 \\ -2 & 6 & 5 & -1 \end{array}\right]$$

$$\xrightarrow{(-1)R_1} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & -2 & 1 & 2 \\ -2 & 6 & 5 & -1 \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & -2 & 1 & 2 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\left(\frac{-1}{2}\right)R_2} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 1 & \frac{-1}{2} & -1 \\ 0 & 2 & -1 & 1 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 1 & \frac{-1}{2} & -1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Back to equation notation:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 1 \\ & & x_2 & - & \frac{1}{2}x_3 & = & -1 \\ & & & & 0 & = & 3 \end{array}$$

- We have a contradiction
  $\implies$ no solution!
- The system is inconsistent.

Recap

Before next class, you should:

- Go over webpage (syllabus, DGDs, grading, etc.).
- Read "Lecture 0" (background on sets, notation, etc.).
- Do recommended exercises.
- Read Sections WILA, SSLE, and RREF of the text.

Next time

- We will develop a systematic procedure for solving systems of equations.
- Our procedure will allow us to solve any system of equations (or determine that it has no solutions).