University of Ottawa – MAT 1302

Network Flow Exercises – Solutions

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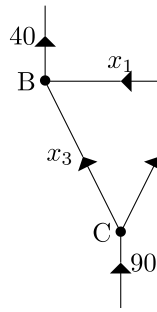
QUESTION 1. Find the general flow pattern of the network shown in the figure below. If you assume that all the flows are nonnegative, what is the largest possible value for x_3?



Solution: We write the equations for each node and the total flow:

Node	**Flow in**		**Flow out**
A	x_2	$=$	$x_1 + x_4$
B	$x_1 + x_3$	$=$	40
C	90	$=$	$x_2 + x_3$
Total	90	$=$	$40 + x_4$

Rearranging the equations into standard form, we get:

$$\begin{array}{ccccccccc} x_1 & - & x_2 & & & + & x_4 & = & 0 \\ x_1 & & & + & x_3 & & & = & 40 \\ & & x_2 & + & x_3 & & & = & 90 \\ & & & & & & x_4 & = & 50 \end{array}$$

We then reduce the augmented matrix:

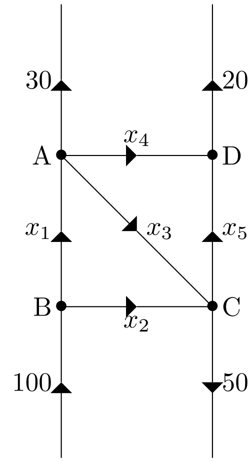
$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 1 & 0 & 90 \\ 0 & 0 & 0 & 1 & 50 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 1 & 0 & 90 \\ 0 & 0 & 0 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is therefore:

$$x_1 = 40 - x_3$$
$$x_2 = 90 - x_3$$
$$x_3 \text{ free}$$
$$x_4 = 50$$

Since x_1 and x_2 cannot be negative, the largest possible value of x_3 is 40.

QUESTION 2. Consider the street network shown below. Flow rates are in cars per minute.



(a) Find the general traffic pattern in the network.

Solution: We write the equations for each node and the total flow:

Node	**Flow in**		**Flow out**
A	x_1	=	$x_3 + x_4 + 30$
B	100	=	$x_1 + x_2$
C	$x_2 + x_3$	=	$x_5 + 50$
D	$x_4 + x_5$	=	20
Total	100	=	100

Rearranging the equations into standard form (and ignoring the last one, which is always satisfied), we get:

$$\begin{aligned} x_1 \quad\quad - x_3 - x_4 \quad\quad &= 30 \\ x_1 + x_2 \quad\quad\quad\quad\quad &= 100 \\ x_2 + x_3 \quad\quad - x_5 &= 50 \\ x_4 + x_5 &= 20 \end{aligned}$$

We then reduce the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 30 \\ 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 1 & 20 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 50 \\ 0 & 1 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 1 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is therefore:

$$\begin{aligned} x_1 &= 50 + x_3 - x_5 \\ x_2 &= 50 - x_3 + x_5 \\ x_4 &= 20 - x_5 \\ x_3, x_5 &\text{ free} \end{aligned}$$

(b) Describe the general traffic pattern with the road whose flow is x_4 is closed.

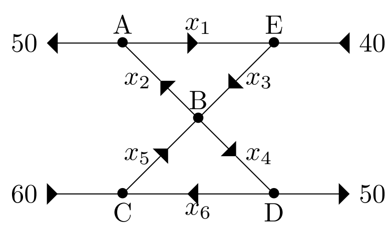
Solution: When this road is closed, $x_4 = 0$. Thus $x_5 = 20$ and so the general solution is:

$$\begin{aligned} x_1 &= 30 + x_3 \\ x_2 &= 70 - x_3 \\ x_4 &= 0 \\ x_5 &= 20 \\ x_3 &\text{ free} \end{aligned}$$

(c) When this road is closed, what is the maximum value of x_3?

Solution: Since all flows must be positive, the maximum value of x_3 is 70.

QUESTION 3. For the network shown below, what are the minimum values of x_2, x_3, x_4, and x_5?



Solution: We write the equations for each node and the total flow:

Node	**Flow in**		**Flow out**
A	x_2	=	$x_1 + 50$
B	$x_3 + x_5$	=	$x_2 + x_4$
C	$x_6 + 60$	=	x_5
D	x_4	=	$x_6 + 50$
E	$x_1 + 40$	=	x_3
Total	100	=	100

Rearranging the equations into standard form (and ignoring the last equation, which is always satisfied), we get:

$$\begin{array}{ccccccccccccc} x_1 & - & x_2 & & & & & & & & & = & -50 \\ & & x_2 & - & x_3 & + & x_4 & - & x_5 & & & = & 0 \\ & & & & & & & & x_5 & - & x_6 & = & 60 \\ & & & & & & x_4 & & & - & x_6 & = & 50 \\ x_1 & & & - & x_3 & & & & & & & = & -40 \end{array}$$

We then reduce the augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is therefore:

$$\begin{aligned} x_1 &= x_3 - 40 \\ x_2 &= x_3 + 10 \\ x_4 &= x_6 + 50 \\ x_5 &= x_6 + 60 \\ x_3, x_6 &\text{ free} \end{aligned}$$

Since x_1 cannot be negative, we must have $x_3 \geq 40$. Thus we must have $x_2 \geq 50$ In addition, x_6 cannot be negative. That implies that $x_4 \geq 50$ and $x_5 \geq 60$. The minimum flows are when $x_1 = x_6 = 0$, in which case $x_2 = 50$, $x_3 = 40$, $x_4 = 50$, $x_5 = 60$.