University of Ottawa – MAT 1302

Markov Chain Exercises – Solutions

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Question 1. (2010 Final Exam) I tend to be rather moody at times. If I am in a good mood today, there is an 80% chance I will still be in a good mood tomorrow and 20% chance that I will be grumpy tomorrow. But if I am grumpy today, there is only a 60% chance that my mood will be good tomorrow.

(a) Determine the migration (transition) matrix for this situation.

Solution: The transition matrix is

$$M = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}.$$

(b) If I am in a good mood on Monday, what is the probability that I will be in a good mood on the coming Wednesday?

Solution: If I am in a good mood on Monday, then the corresponding probability vector is $\vec{p}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For Tuesday, we have $\vec{p}_1 = M\vec{p}_0 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$.

On Wednesday, $\vec{p}_2 = M\vec{p}_1 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix}$. Therefore, there is a 76% chance that I will be in a good mood on Wednesday.

(c) In the long term, what percentage/fraction of the time am I in a good mood? *Hint:* Find the steady-state vector of the migration matrix and justify why it describes the long term behaviour.

Solution: Since the matrix M is regular stochastic, the long term behaviour is given by the steady-state vector. To find the steady-state vector, solve $(M - I)\vec{x} = \vec{0}$.

$$M - I = \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix}.$$

To find x_2 solve $3x_2 + x_2 = 1$ to get $x_2 = 1/4$. Therefore the steady-state vector is

$$\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}.$$

The percentage of time that I am in a good mood is 75% or the fraction of time that I am in a good mood is $\frac{3}{4}$.

Question 2. (2009 Final Exam) Two telephone companies *OutOfOrder* and *NoService* are competing. A statistical study has shown that in each **6 month** period, 60% of the clients of OutOfOrder stay with the company while 40% of them which to NoService. During the same period, 20% of the clients of NoService switch to OutOfOrder, while 80% stay with NoService. On January 1, 2009, OutOfOrder had 50 thousand clients and NoService had 25 thousand clients.

(a) Write the migration matrix M.

Solution:

$$M = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}.$$

(b) How many clients will each company have on July 1, 2009?

Solution:

$$\begin{aligned} \vec{x}_1 &= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 50\,000 \\ 25\,000 \end{bmatrix} \\ &= \begin{bmatrix} 35\,000 \\ 40\,000 \end{bmatrix}. \end{aligned}$$

Thus, OutOfOrder will have 35 000 customers and NoService will have 40 000 customers.

(c) How many clients will each company have on January 1, 2010?

Solution:

$$\begin{aligned} \vec{x}_2 &= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 35\,000 \\ 40\,000 \end{bmatrix} \\ &= \begin{bmatrix} 29\,000 \\ 46\,000 \end{bmatrix}. \end{aligned}$$

Thus, OutOfOrder will have 29 000 customers and NoService will have 46 000 customers.

(d) Assuming the migration matrix M stays constant in the long term, find the market share (i.e. percentage of the total customers) of each company in the long term. That is, find the equilibrium market shares.

Solution: Since M is a regular stochastic matrix, the equilibrium is given by an eigenvector with eigenvalue one. So we must solve

$$M\vec{q} = \vec{q} \qquad \text{or} \qquad (M - I)\vec{q} = \vec{q}.$$

We row reduce:

$$M - I = \begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -0.5 \\ 0 & 0 \end{bmatrix}.$$

Thus, a basis for the eigenspace is $\{(1, 2)\}$. Thus, there exists a scalar $\alpha \neq 0$ such that

$$(q_1, q_2) = \alpha(1, 2).$$

Since the market shares must add up to one, we have

$$1 = q_1 + q_2 = \alpha(1 + 2) = 3\alpha \implies \alpha = \frac{1}{3}.$$

Therefore

$$\vec{q} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

Thus, in the long term, OutOfOrder has 1/3 of the market and NoService has 2/3 of the market.

Question 3. (2008 Final Exam) In a certain country, the mobile phone industry is dominated by two companies: Ten-Fours and Siren. Ten-Fours has 180 000 customers and Siren has 120 000 customers. Assume that, every year, 10% of the customer base of Ten-Fours switches to Siren and 5% of the customer base of Siren switches to Ten-Fours. For the purposes of this question, suppose no customer leaves a company without switching to the other one and no company attracts customers that are not leaving the other (that is, the only changes in customer base come from switching between the two companies).

(a) Write down the transition (migration) matrix M and initial state vector $\mathbf{x}_0$ for this problem.

Solution:

$$M = \begin{bmatrix} .9 & .05 \\ .1 & .95 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 180\,000 \\ 120\,000 \end{bmatrix}.$$

(b) Find the number of customers of Ten-Fours after one year.

Solution:

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .9 & .05 \\ .1 & .95 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .56 \\ .44 \end{bmatrix}$$

Therefore, after one year, Ten-Fours will have $(0.56)(300\,000) = 168\,000$ customers.

(c) Find the number of customers of Ten-Fours after many years. That is, find the long term stable number of customers of Ten-Fours.

Solution: Since M is a regular stochastic matrix (all of its entries are strictly positive), the state will approach the unique steady-state vector $\mathbf{q}$. To find $\mathbf{q}$, we find the eigenspace with eigenvalue 1.

$$\left[\ (M - I) \mid \mathbf{0}\ \right] = \left[\begin{array}{cc|c} -.1 & .05 & 0 \\ .1 & -.05 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

Since we want $\mathbf{q}$ to be a probability vector, we choose x_2 so that the entries sum to one. So

$$x_3 = \left(\frac{1}{2} + 1\right)^{-1} = \frac{2}{3}.$$

Therefore

$$\mathbf{q} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$

Thus, in the long term, Ten-Fours has $\frac{1}{3}(300\,000) = 100\,000$ customers.