MAT 1302 – MARKOV CHAIN EXAMPLE
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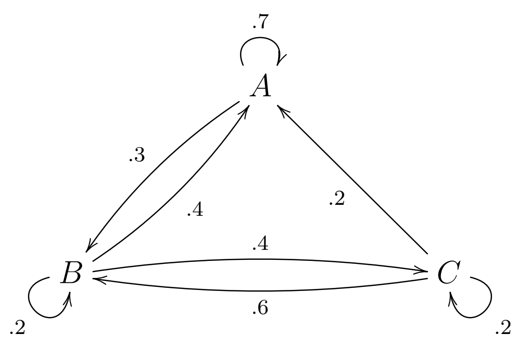
Suppose a city has 3 internet service providers (A, B, and C). A starts with 200 000 customers and B and C each start with 400 000 customers. Suppose that, each year, the following migration occurs



How many customers does each company have

(a) after one year?
(b) after two years?
(c) after many years?

Solution:
(a). The migration matrix is

$$M = \begin{bmatrix} .7 & .4 & .2 \\ .3 & .2 & .6 \\ 0 & .4 & .2 \end{bmatrix}$$

and the initial state vector is

$$\mathbf{x}_0 = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .7 & .4 & .2 \\ .3 & .2 & .6 \\ 0 & .4 & .2 \end{bmatrix} \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix} = \begin{bmatrix} 0.38 \\ 0.38 \\ 0.24 \end{bmatrix}.$$

So after one year, A and B have 380 000 customers each and C has 240 000 customers.

(b).

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .7 & .4 & .2 \\ .3 & .2 & .6 \\ 0 & .4 & .2 \end{bmatrix} \begin{bmatrix} .38 \\ .38 \\ .24 \end{bmatrix} = \begin{bmatrix} .466 \\ .334 \\ .2 \end{bmatrix}.$$

So after two years, A has 466 000 customers, B has 334 00 customers and C has 200 000 customers.

(c).

$$M^2 = \begin{bmatrix} .61 & .41 & .42 \\ .27 & .4 & .3 \\ .12 & .16 & .28 \end{bmatrix}$$

Since all the entries of M^2 are strictly greater than zero, M is a regular stochastic matrix (we needed to check higher powers of M since M itself had a zero entry). Therefore $\mathbf{x}_k$ approaches the unique steady-state vector $\mathbf{q}$ as $k \to \infty$. To find the steady state vector, we solve

$$M\mathbf{q} = \mathbf{q} \Longleftrightarrow (M - I)\mathbf{q} = \mathbf{0}$$

which amounts to row reducing

$$\left[\ (M-I) \mid \mathbf{0}\ \right] \sim \left[\begin{array}{ccc|c} -.3 & .4 & .2 & 0 \\ .3 & -.8 & .6 & 0 \\ 0 & .4 & -.8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & 4 & 2 & 0 \\ 3 & -8 & 6 & 0 \\ 0 & 4 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{10}{3} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is

$$\begin{aligned} x_1 &= \frac{10}{3}x_3 \\ x_2 &= 2x_3 \\ x_3 &\quad \text{free} \end{aligned}$$

Switching to vector notation gives

$$\mathbf{x} = x_3 \begin{bmatrix} 10/3 \\ 2 \\ 1 \end{bmatrix}.$$

Any choice of $x_3 \neq 0$ gives an eigenvector of M with eigenvalue 1. However, we want to choose x_3 so that the resulting vector is a probability vector (that is, its entries add to one). So we pick x_3 to be the reciprocal of the sum of the entries

$$x_3 = \left(\frac{10}{3} + 2 + 1\right)^{-1} = \left(\frac{19}{3}\right)^{-1} = \frac{3}{19}.$$

Therefore

$$\mathbf{q} = \frac{3}{19}\begin{bmatrix} 10/3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10/19 \\ 6/19 \\ 3/19 \end{bmatrix}.$$

Thus, in the long term A has $\frac{10}{19}(1\,000\,000) \cong 526\,316$ customers, B has $\frac{6}{19}(1\,000\,000) \cong 315\,789$ customers, and C has $\frac{3}{19}(1,\ 000\,000) \cong 157\,895$ customers.