University of Ottawa – MAT 1302

Leontief Input-Output Model Exercises – Solutions

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QUESTION 1. Suppose an economy has three sectors. The consumption matrix and final demand vector are given by

$$C = \begin{bmatrix} .4 & .6 & 0 \\ .4 & .3 & .2 \\ .1 & 0 & .4 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 72 \\ 150 \\ 58 \end{bmatrix}.$$

Using the Leontief Input-Output Model, determine the production levels necessary to satisfy the final demand.

Solution: We need to solve $\vec{x} = C\vec{x} + \vec{d}$, or equivalently

$$(I - C)\vec{x} = \vec{d}.$$

We row reduce the augmented matrix:

$$\left[\begin{array}{c|c} I - C & \vec{d} \end{array}\right] = \left[\begin{array}{ccc|c} .6 & -.6 & 0 & 72 \\ -.4 & .7 & -.2 & 150 \\ -.1 & 0 & .6 & 58 \end{array}\right] \xrightarrow[10R_3]{\substack{10R_1 \\ 10R_2}} \left[\begin{array}{ccc|c} 6 & -6 & 0 & 720 \\ -4 & 7 & -2 & 1500 \\ -1 & 0 & 6 & 580 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 120 \\ -4 & 7 & -2 & 1500 \\ -1 & 0 & 6 & 580 \end{array}\right] \xrightarrow[R_1+R_3]{4R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 120 \\ 0 & 3 & -2 & 1980 \\ 0 & -1 & 6 & 700 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 120 \\ 0 & 1 & -2/3 & 660 \\ 0 & -1 & 6 & 700 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 780 \\ 0 & 1 & -2/3 & 660 \\ 0 & 0 & 16/3 & 1360 \end{array}\right] \xrightarrow{\frac{3}{16}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 780 \\ 0 & 1 & -2/3 & 660 \\ 0 & 0 & 1 & 255 \end{array}\right] \xrightarrow[\frac{2}{3}R_3+R_2]{\frac{2}{3}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 950 \\ 0 & 1 & 0 & 830 \\ 0 & 0 & 1 & 255 \end{array}\right]$$

Thus the production level necessary to meet the final demand is

$$\vec{x} = \begin{bmatrix} 950 \\ 830 \\ 255 \end{bmatrix}.$$

QUESTION 2. Suppose that an economy has two sectors, Mining and Electricity. For each unit of output, Mining requires 0.4 units of its own production and 0.2 units of Electricity. Moreover, for each unit of output, Electricity requires 0.2 units of Mining and 0.6 units of its own production.

(a) Determine the consumption matrix C for this economy.

Solution: $C = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}$.

(b) Find the inverse of $(I - C)$.

Solution:

$$I - C = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 6/10 & -2/10 \\ -2/10 & 4/10 \end{bmatrix} \implies$$

$$(I - C)^{-1} = \frac{1}{(\frac{6}{10})(\frac{4}{10}) - (\frac{-2}{10})(\frac{-2}{10})} \begin{bmatrix} 4/10 & 2/10 \\ 2/10 & 6/10 \end{bmatrix} = (5) \begin{bmatrix} 4/10 & 2/10 \\ 2/10 & 6/10 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

(c) Using the Leontief Model, determine the production levels from each sector that are necessary to satisfy a final demand of 20 units from Mining and 10 units from Electricity. Use the inverse of $(I - C)$ in your calculation (that is, use the inverse matrix method to solve this problem).

Solution: Let $\vec{d} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$. The Leontief Model states that the product vector $\vec{x}$ must satisfy $C\vec{x} + \vec{d} = \vec{x}$, or equivalently

$$\vec{x} = (I - C)^{-1}\vec{d} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}.$$

Thus, the production level needed to meet the demand $\vec{d}$ is $\begin{bmatrix} 50 \\ 50 \end{bmatrix}$, or 50 units from Mining and 50 units from Electricity.

QUESTION 3. An economy has two sectors: Electricity and Services. For each unit of output, Electricity requires 0.5 units from its own sector and 0.4 units from Services. Meanwhile, Services requires 0.5 units from Electricity and 0.2 units from its own sector to produce one unit of Services.

(a) Determine the consumption matrix C.

Solution: $C = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.2 \end{bmatrix}$.

(b) State the Leontief input-output equation relating C to the production vector $\vec{x}$ and final demand vector $\vec{d}$.

Solution: $\vec{x} = C\vec{x} + \vec{d}$.

(c) Use an *inverse matrix* to determine the production vector necessary to satisfy a final demand of 1000 units of Electricity and 2000 units of Services, i.e. $\vec{d} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$.

Solution: We need to solve $(I - C)\vec{x} = \vec{d}$.

$$I - C = \begin{bmatrix} 0.5 & -0.5 \\ -0.4 & 0.8 \end{bmatrix} \Rightarrow \det(I - C) = (0.5)(0.8) - (-0.5)(-0.4) = 0.2,$$

$$(I - C)^{-1} = \frac{1}{0.2} \begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} = 5 \begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} = \begin{bmatrix} 4 & 2.5 \\ 2 & 2.5 \end{bmatrix}$$

Therefore,

$$\vec{x} = (I - C)^{-1}\vec{d} = \begin{bmatrix} 4 & 2.5 \\ 2 & 2.5 \end{bmatrix} \begin{bmatrix} 1000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 9000 \\ 7000 \end{bmatrix}.$$