Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

MAT 1302A: Mathematical Methods II

Professor : Aaron Tikuisis

Midterm 3 – V.A.

March 22, 2019

Name ________________________ Surname ______________________

Student # ________________________ DGD ______________________

Instructions:

- Write your full name and student number at the top of each page in the designated space.
- The duration of this midterm is 70 minutes.
- This is a closed book midterm that contains **6 questions**.
- No calculators are permitted.
- Use the designated space to answer each question. If ever you need additional space, you may write on the back of the page. If you do so, you must clearly indicate where the rest of your work can be found.
- Unless otherwise specified, the correct answer requires full justification written legibly and logically: you must convince me that you know your solution is correct.
- There is an additional blank page at the end of this exam that you may use as scrap paper.
- The use of cellphones, electronic devices and course notes is strictly forbidden. All phones and electronic devices must be turned off and kept in your bags: do not leave them on you. If you are seen to have an electronic device on your person, we may ask you to leave the exam immediately, and fraud allegations could be made, which could lead to a mark of 0 (zero) on this midterm.

Good luck!!!

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ____________________________

Do not write in the following table.

Question	1	2	3	4	5	6	Total
Maximum	3	2	4	3	3	5	20
Note							

Question 1. [**3 points**] Let A be an $n \times n$ invertible matrix. Answer **TRUE** or **FALSE** for each of the following. No justification is required.

(a) The rank of A is $n-1$. **Answer:** **Solution: F**

(b) $\mathcal{N}(A) = \{\mathbf{0}\}$. **Answer:** **Solution: T**

(c) A^t is not invertible. **Answer:** **Solution: F**

(d) The columns of A are linearly dependent. **Answer:** **Solution: F**

(e) $\det(A) \neq 0$. **Answer:** **Solution: T**

(f) The rows of A form a basis of $\mathbb{R}^n$. **Answer:** **Solution: T**

(g) $\mathcal{C}(A) = \{\mathbf{0}\}$. **Answer:** **Solution: F**

(h) $\det(A^t A^{-1}) = 1$. **Answer:** **Solution: T**

-0.5 for each wrong answer, with minimum 0 point

Question 2. [**2 points**] Let A, B and C be 3×3 matrices such that

$$\det A = 3, \quad \det B = -2, \quad \det\left(-2A^tB^{-1}CB\right) = 48.$$

Find $\det C$.

Solution:

$$\begin{aligned} 48 &= \det\left(-2A^tB^{-1}CB\right) = (-2)^3 \cdot \det(A^t) \cdot \det(B^{-1}) \cdot \det(C) \cdot \det(B) \\ &= (-8) \cdot \det(A) \cdot \tfrac{1}{\det(B)} \cdot \det(C) \cdot \det(B) = (-8) \cdot 3 \cdot \det(C) = (-24) \cdot \det(C) \end{aligned}$$

So $\det(C) = \frac{48}{-24} = -2$.

- 0.5 for noticing that $\det(2\ldots) = (-2)^3 \det(\ldots)$.
- 0.5 for decomposing the determinant.
- 0.5 for realizing how to compute $\det(C)$ in terms of the given quantities and the remaining steps of the computation.
- 0.5 for the correct final answer.

Question 3. [**4 points**] Let $A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & 2 & 0 & 1 \\ -2 & 3 & 0 & -2 \\ 2 & 0 & -1 & 1 \end{bmatrix}$.

Find $\det(A)$ using the recursive formula.

Solution:

$$\begin{aligned}
\begin{vmatrix} 1 & 0 & -3 & 0 \\ 2 & 2 & 0 & 1 \\ -2 & 3 & 0 & -2 \\ 2 & 0 & -1 & 1 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 2 & 0 & 1 \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{vmatrix} + (-3) \cdot \begin{vmatrix} 2 & 2 & 1 \\ -2 & 3 & -2 \\ 2 & 0 & 1 \end{vmatrix} \\
&= -(-1) \cdot \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + (-3) \cdot \left(2 \cdot \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} \right) \\
&= 1 \cdot (-7) + (-3) \cdot [2 \cdot (-7) + 1 \cdot 10] \\
&= -7 + (-3)[-14 + 10] \\
&= -7 + 12 \\
&= 5.
\end{aligned}$$

- 0.5 for breaking it into 2 determinants.
- 1 pt for each 3×3 determinant.
- 0.5 for the correct answer

Question 4. Consider the complex numbers $z = 2 - i$ and $w = 1 + i$. Express the following in the form $a + bi$ $(a, b \in \mathbb{R})$.

(a) **[1 point]** $z - 2w$

Solution:

$$z - 2w = (2 - i) - 2(1 + i) = 2 - i - 2 - 2i = 0 - 3i = -3i.$$

0.5 pt for correct real part; 0.5 pt for correct imaginary part.

(b) **[2 points]** $\dfrac{w}{\bar{z}}$

Solution:

$$\begin{aligned} \frac{w}{\bar{z}} &= \frac{1+i}{2+i} = \frac{(1+i)(2-i)}{(2+i)(2-i)} = \frac{2 - i + 2i + 1}{5} \\ &= \frac{3+i}{5} = \frac{3}{5} + \frac{1}{5}i \end{aligned}$$

- 0.5 pt for the conjugate.
- 0.5 pt for multiplying by z.
- 1 pt for the rest.

Question 5. [**3 points**] For each of the following subsets of $\mathbb{R}^3$, answer (**YES** or **NO**) whether the subset is a subspace of $\mathbb{R}^3$. No justification is required.

(a) $T = \left\{ \begin{bmatrix} a-2b \\ b-3a \\ -3b \end{bmatrix} \middle| a,\, b \in \mathbb{R} \right\}$ **Answer:** **Solution:** Y

Solution:

$$T = \text{span}\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \right\}$$

(b) $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x+y=1 \in \mathbb{R} \right\}$ **Answer:** **Solution:** N

Solution: For example, the zero vector does not belong to U.

(c) $V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ **Answer:** **Solution:** N

Solution: For example, the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in V, but the scalar multiple $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ is not.

(d) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ **Answer:** **Solution:** Y

Solution: W is the null space of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

(e) $X = \left\{ \begin{bmatrix} 2 & 0 & 1 \\ 2 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$ **Answer:** **Solution:** Y

Solution: X is the column space of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 3 & 9 \end{bmatrix}$.

(f) $Y = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x^2 - 2yz = 0 \right\}$ **Answer:** **Solution:** N

Solution: For example, the vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is in Y, but the scalar multiple $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ is not.

- 0.5 for each correct answer.

Question 6. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 2 & 2 & 1 & -2 \\ 2 & 2 & 0 & 0 \end{bmatrix}$, whose RREF is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) [**2 points**] Give a basis for $\mathcal{C}(A)$ and its dimension.

Solution: A basis for $\mathcal{C}(A)$ is given by the pivot columns of A, which are column 1 and column 3. Therefore the set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathcal{C}(A)$. The dimension is 2.

1 point for a correct basis.
1 point for correct rank.

(b) [**2 points**] Find a basis for $\mathcal{N}(A)$.

Solution: We first write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

A basis for $\mathrm{Nul}(A)$ is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

1 pt for the correct parametric vector form of the solution.
1 pt for the basis.

(c) [**1 point**] If B is a 1302×2019 matrix such that $\mathrm{rank}(B) = 1000$, what is $\dim(\mathcal{N}(A))$?

Solution: By the rank theorem,

$$\mathrm{rank}(B) + \dim(\mathcal{N}(B)) = 2019.$$

Therefore

$$\dim(\mathcal{N}(B)) = 2019 - 1000 = 1019.$$

1 pt; all or nothing

Extra page (V-A)