University of Ottawa

Department of Mathematics and Statistics

MAT 1302A : Mathematical Methods II

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Third Midterm Exam – Version A

March 23, 2018

Last Name________________________ First Name ________________________

Student Number # ________________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ________________________

Please do not write anything in the following table.

Question	1	2	3	4	5	6	Total
Maximum	2	3	2	4	6	3	20
Points							

1. [**2 points**] For each of the following sets, write "Yes" if the set is a subspace of $\mathbb{R}^n$ for the given value of n, and write "No" if it is not a subspace of $\mathbb{R}^n$. You will receive .5 points for each correct answer, and receive -.5 points for each incorrect answer. You will not receive a negative total score on this question.

_____ A line in $\mathbb{R}^2$ that does not pass through the origin, $n = 2$.

Solution: Non

_____ The set $\left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \;\middle|\; x, y, z, t \in \mathbb{R} \text{ and } 2x - 3t = 2y - \frac{1}{2}z \right\}$, $n = 4$.

Solution: Oui

_____ The set $\left\{ \begin{bmatrix} x - y \\ x + y \\ 2y \end{bmatrix} \;\middle|\; x,\, y \in \mathbb{R} \right\}$, $n = 3$

Solution: Oui

_____ The set $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x,\, y,\, z \in \mathbb{R} \text{ and } z = 1 + x \right\}$, $n = 3$.

Solution: Non

2. [**3 points**] For each of the following statements write "**True**" if the statement is true, and write "**False**" if the statement is false. You will receive .5 points for each correct answer, and receive -.5 points for each incorrect answer. You will not receive a negative total score on this question.

_____ For every pair of 2×2 matrices A and B we have $\det(A - B) = \det A - \det B$.

Solution: False

_____ For every $n \times n$ matrix A we have $\det A = -\det(A^T)$.

Solution: False

_____ If an $n \times n$ matrix A is invertible then $\operatorname{Nul} A = \{0\}$.

Solution: True

_____ If A is a 3×3 matrix such that two of its columns are equal, then we have $\det A = 0$.

Solution: True

_____ The dimension of $\operatorname{Nul} A$ is always equal to the number of pivot columns of A.

Solution: False

_____ The columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$.

Solution: True

3. [**2 points**] Using the method of cofactor expansion compute the determinant of the matrix

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & -3 \\ 43 & 4 & 0 & 17 & -8 \\ 28 & 0 & 0 & 5 & -4 \\ 18 & 5 & -3 & 0 & 9 \\ 4 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Solution:

By expansion with respect to the 3rd column we obtain:

$$\det A = 3 \begin{vmatrix} -2 & 0 & 0 & -3 \\ 43 & 4 & 17 & -8 \\ 28 & 0 & 5 & -4 \\ 4 & 0 & 0 & 3 \end{vmatrix}.$$

Next, we expand with respect to the second column and obtain

$$\det A = (3)(4) \begin{vmatrix} -2 & 0 & -3 \\ 28 & 5 & -4 \\ 4 & 0 & 3 \end{vmatrix}.$$

Next, we expand with respect to the 2nd column and obtain

$$\det A = (3)(4)(5) \begin{vmatrix} -2 & -3 \\ 4 & 3 \end{vmatrix}.$$

Finally, we have:

$$\det A = (3)(4)(5)(-6 + 12) = 360.$$

4. (a) Suppose that A, B and C are 4×4 matrices such that $\det A = -\frac{1}{2}$, $\det B = -1$ and $\det C = 3$. Also, suppose that $X = -A^T(C^T)^{-1}B^3C$.

i) **[2 points]** Calculate $\det X$

Solution: We have

$$\begin{aligned}\det X &= \det\left(-A^T(C^T)^{-1}B^3C\right) = (-1)^4 \det(A^T)\,\det(C^T)^{-1}\,\det B^3\,\det C \\ &= \det A\,(\det C)^{-1}\,\det B^3\,\det C = \det A\,(\det B)^3 = \frac{-1}{2}\,(-1)^3 = \frac{1}{2}\end{aligned}$$

ii) **[0.5 points]** Is X invertible? You should justify your answer.

Solution: Yes, X is invetible because $\det(X) \neq 0$.

(b) **[1.5 points]** Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a 3×3 matrix such that $\det A = -5$. Compute $\det B$, where $B = \begin{bmatrix} a & b & c \\ 5g & 5h & 5i \\ -d & -e & -f \end{bmatrix}$. You should **clearly** identify the elementary row operations that you use in your calculation.

Solution: The matrix B is obtained from A by the row operations $R_2 \to -R_2$, $R_3 \to 5R_3$ and $R_3 \leftrightarrow R_2$. Thus $\det B = (-1)(5)(-1)\det A = -25$.

5. Let $A = \begin{bmatrix} -1 & 0 & -1 & 1 & -2 \\ 2 & -\frac{1}{2} & 2 & 0 & 3 \\ 3 & -1 & 3 & 1 & 4 \end{bmatrix}$.

a) [**3 points**] Find a basis for Col A.

Solution:

$$\begin{bmatrix} -1 & 0 & -1 & 1 & -2 \\ 2 & -\frac{1}{2} & 2 & 0 & 3 \\ 3 & -1 & 3 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 2 & -\frac{1}{2} & 2 & 0 & 3 \\ 3 & -1 & 3 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & -\frac{1}{2} & 0 & 2 & -1 \\ 0 & -1 & 0 & 4 & -2 \end{bmatrix} \xrightarrow{R_2 \to -2R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & -4 & 2 \\ 0 & -1 & 0 & 4 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and second columns of A are pivot columns. Therefore they form a basis for Col A. Thus the basis for Col A is given by $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -1 \end{bmatrix} \right\}$

b) [**3 points**] Find a basis for Nul A.

Solution: Using the R.E.F. computed in part (a), we obtain the vector parametric form of the general solution:

$$\left\{ \begin{array}{rcl} x_1 & = & -x_3 + x_4 - 2x_5 \\ x_2 & = & 4x_4 - 2x_5 \end{array} \right. \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Consequently, a basis for Nul A consists of $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

6. [**3 points**] Suppose that $z = 1 - 3i$ and $w = 5 + i$. Compute $z\overline{w}$ and $\dfrac{w}{\overline{z}}$. Write your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

Solution: On a

$$z\overline{w} = (1-3i)\,\overline{5+i} = (1-3i)(5-i) = 5 - i - 15i - 3 = 2 - 16i$$

et

$$\frac{w}{\overline{z}} = \frac{5+i}{\overline{1-3i}} = \frac{5+i}{1+3i} = (5+i)\frac{1-3i}{9+1} = \frac{5-15i+i+3}{9+1} = \frac{8-14i}{10} = \frac{4}{5} - \frac{7}{5}i.$$