University of Ottawa

Department of Mathematics and Statistics

MAT 1302A : Mathematical Methods II
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Second Midterm Exam – Version A

March 2, 2018

Surname ______________________ First Name ______________________

Student # ______________________ DGD ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ______________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	4	4	5	3	4	24
Grade							

1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also, let I_2 denote the 2×2 identity matrix.

(a) [**1 point**] Among the four expressions below, only one makes sense. Circle it (no justification is needed).

- $AB - 2I_2$
- $(A+B)^{-1}$
- $I_2 - A^T AB^T$
- $(B^T A - I_2)^{-1}$

Solution:

- In the first expression, A and B are 3×2 matrices and so AB does not make sense.
- The second expression does not make sense since $A + B$ is a 3×2 matrix, and so it is not invertible.
- In the third expression, $A^T AB^T$ is a 2×3 matrix. It does not make sense to subtract from it a 2×2 matrix.
- The fourth expression makes perfect sense.

(b) [**3 points**] Calculate explicitly the above expression which makes sense.

Solution:

$$B^T = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

So

$$B^T A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -3 & 5 \end{bmatrix}$$

Thus

$$B^T A - I_2 = \begin{bmatrix} 2 & 2 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The determinant of $B^T A - I_2$ is $1 \cdot 4 - 2 \cdot (-3) = 10$. Therefore, we have

$$(B^T A - I_2)^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

2. [**4 points**] For each of the following statements, indicate if it is true or false. You will receive 1 point for every correct answer and lose .5 points for every incorrect answer (but you cannot receive a negative mark on this question).

_____ For an invertible $n \times n$ matrix A and a vector $\mathbf{b}$ in $\mathbb{R}^n$, the linear system $A\mathbf{x} = \mathbf{b}$ always has a unique solution.

Solution: True.

_____ For two $n \times n$ matrices A and B, we always have $(AB)^T = A^T B^T$

Solution: False.

_____ For every two $n \times n$ matrices A and B, if $AB = 0$ and $B \neq 0$, then $A = 0$.

Solution: False.

_____ The vectors $\left\{ \begin{bmatrix} 1.4 \\ 0.6 \\ -2.9 \\ -7.1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 40 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 3.9 \\ 21 \end{bmatrix}, \begin{bmatrix} 0 \\ -4.1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 3.2 \\ 5.7 \end{bmatrix} \right\}$ are linearly independent.

Solution: False.

3. [**4 points**] Is the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

invertible? If the answer is yes, determine its inverse.

Solution: We row-reduce the super-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 6 & -3 & -2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 6 & 4 \\ 0 & 1 & 0 & 6 & -3 & -2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

We conclude that the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -11 & 6 & 4 \\ 6 & -3 & -2 \\ -2 & 1 & 1 \end{bmatrix}.$$

4. An economy consists of two sectors: Agriculture and Industry. In order to produce 1 unit of output, the Agriculture sector uses $\frac{2}{5}$ units from the Agriculture and $\frac{1}{5}$ units from the Industry sector. Further, to produce 1 unit of output, the Industry sector uses $\frac{3}{5}$ units from the Agriculture and $\frac{1}{5}$ units from the Industry sector.

(a) [**1 point**] Write down the consumption matrix C for this economy.

Solution:

$$C = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

(b) [**1 point**] What intermediate demands are created if Agriculture is supposed to produce 15 units, and simultaneously Industry is supposed to produce 25 units? Write down your final answers in the blank spots below.

The intermediate demands are _____ units from Agriculture and _____ units from Industry.

Solution:

$$C \begin{bmatrix} 15 \\ 25 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 15 \\ 25 \end{bmatrix} = \begin{bmatrix} 21 \\ 8 \end{bmatrix}$$

The intermediate demands are 21 units from Agriculture and 8 units from Industry.

(c) [**3 points**] Determine the production levels required to meet a final demand of 9 units from the Agriculture sector and 15 units from the Industry sector. Write down your final answers in the blank spots below.

The required production levels are _____ units from Agriculture and _____ units from Industry.

Solution: We should solve the equation $A\mathbf{x} = \mathbf{d}$ where

$$A = I_2 - C = \begin{bmatrix} \frac{3}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} \frac{3}{5} & -\frac{3}{5} & 9 \\ -\frac{1}{5} & \frac{4}{5} & 15 \end{array}\right] \xrightarrow[R_2 \to 5R_2]{R_1 \to \frac{5}{3}R_1} \left[\begin{array}{cc|c} 1 & -1 & 15 \\ -1 & 4 & 75 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & -1 & 15 \\ 0 & 3 & 90 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & -1 & 15 \\ 0 & 1 & 30 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 45 \\ 0 & 1 & 30 \end{array}\right]$$

Therefore $\mathbf{x} = \begin{bmatrix} 45 \\ 30 \end{bmatrix}$. The required production levels are 45 units from Agriculture and 30 units from Industry.

5. [**3 points**] Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ -1 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 6 \\ 2 \end{bmatrix}$$

Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ linearly independent? If they are linearly dependent, then write down a linear dependence relation for these vectors. You should justify your answers.

Solution: We form the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and calculate an echelon form:

$$\begin{bmatrix} 2 & -4 & 6 \\ -1 & -1 & 6 \\ 3 & -4 & 2 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & 6 \\ 3 & -4 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 9 \\ 3 & -4 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 9 \\ 0 & 2 & -7 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{3} R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 2 & -7 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

Since all columns are pivot columns, the homogeneous linear system with coefficient matrix A has a unique solution (the trivial one). Therefore the given vectors are linearly independent.

6. [**4 points**] Let A and B be two $n \times n$ invertible matrices. Find the $n \times n$ matrix X in terms of A and B such that

$$AB^{-1}(X+B)^T(BA)^{-1} - AB = 0$$

Solution:

$$\begin{aligned}
& AB^{-1}(X+B)^T(BA)^{-1} - AB = 0 \\
\Longrightarrow\ & AB^{-1}(X+B)^T(BA)^{-1} = AB \\
\Longrightarrow\ & B^{-1}(X+B)^T = A^{-1}AB(BA) \\
\Longrightarrow\ & B^{-1}(X+B)^T = B^2A \\
\Longrightarrow\ & (X+B)^T = B^3A \\
\Longrightarrow\ & X+B = (B^3A)^T \\
\Longrightarrow\ & X+B = A^T(B^3)^T \\
\Longrightarrow\ & X = A^T(B^3)^T - B
\end{aligned}$$