University of Ottawa

Department of Mathematics and Statistics

MAT 1302A : Mathematical Methods II
Professor: Hadi Salmasian

First Midterm Exam – Version A

February 2, 2018

Surname ____________________ First Name ____________________

Student # ____________________ DGD ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ____________________

Please do not write in the table below.

Question	1	2	3	4	5	6	7	Total
Maximum	4	3	2	4	4	4	2	23
Grade								

1. [**4 points**] Write down the general solution of the following linear system in vector parametric form.

$$\begin{cases} x & -3y & +z & +2t & = 2 \\ & & z & +2t & = -1 \\ -x & +3y & -2z & -4t & = -1 \\ 2x & -6y & +4z & +8t & = 2 \end{cases}$$

Solution: The R.E.F of the the augmented matrix is obtained as follows:

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ -1 & 3 & -2 & -4 & -1 \\ 2 & -6 & 4 & 8 & 2 \end{array}\right] \xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 + R_1} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 2 & 4 & -2 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{E.F.}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{R.E.F.}$$

From the E.F. or the R.E.F. it is clear that the system is consistent. The pivot columns are the first and the third columns, so x and z are basic, and y and t are free. The R.E.F. is the augmented matrix of the linear system

$$\begin{cases} x & -3y & & & = & 3 \\ & & z & +2t & = & -1 \end{cases}$$

The general solution is

$$\begin{cases} x & = 3y + 3 \\ y & : \text{free} \\ z & = -2t - 1 \\ t & : \text{free} \end{cases}$$

In vector parametric form, the solution is

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 3y+3 \\ y \\ -2t-1 \\ t \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

2. [**3 points**] Determine all values of the parameter h such that the linear system

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & h & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

is consistent. You should justify your answer.

Solution: The augmented matrix of this system can be reduced as follows

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & h & 2 & 3 \\ 2 & -2 & 3 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & h+2 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

If $h + 2 = 0$ then the rightmost column will be a pivot column (look at the second row!) and therefore the system will be inconsistent. However, if $h + 2 \neq 0$, then the matrix above will be in E.F. and the rightmost column will not be a pivot column. Hence the system will be consistent. Therefore the answer is all $h \neq -2$.

3. Compute the following:

(a) [**1 point**] $\begin{bmatrix} -1 & 3 \\ \frac{1}{2} & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Solution: $\begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

(b) [**1 point**] $-3\mathbf{a} + 2A\mathbf{b}$ where $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Solution: $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$

4. (a) **[2 points]** Let A be a 4×4 matrix, and set $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -\frac{1}{3} \end{bmatrix}$, and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

We know that the general solution of the linear system $A\mathbf{x} = \mathbf{b}$ is:

$$\begin{cases} x_1 & = -3x_2 + \frac{1}{3}x_4 - 3 \\ x_2 & : \text{free} \\ x_3 & = -2x_4 - \frac{1}{2} \\ x_4 & : \text{free} \end{cases}$$

Is the linear system $A\mathbf{x} = \mathbf{0}$ consistent? If your answer is "Yes", you should write down the general solution of the system $A\mathbf{x} = \mathbf{0}$.

Solution: The system $A\mathbf{x} = \mathbf{0}$ is homogeneous, therefore it is consistent. From the relation between the solution of the systems $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ it follows that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 & = -3x_2 + \frac{1}{3}x_4 \\ x_2 & : \text{free} \\ x_3 & = -2x_4 \\ x_4 & : \text{free} \end{cases}$$

(b) **[2 points]** Is it true that for any choice of b_1, b_2, b_3 in $\mathbb{R}$, the linear system

$$\begin{cases} x_1 & & -2x_3 & +3x_4 & -x_5 & = & b_1 \\ & -x_2 & +x_3 & & +5x_5 & = & b_2 \\ 2x_1 & -x_2 & -3x_3 & +6x_4 & +3x_5 & = & b_3 \end{cases}$$

is consistent? You should justify your answer.

Solution: No, because the E.F. of the coefficient matrix is

$$\begin{bmatrix} 1 & 0 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 & 5 \\ 2 & -1 & -3 & 6 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ E.F.}$$

The 3rd row does not contain any pivot positions. Therefore by the criterion for "Guaranteed Existence of Solution", the statement is not true.

5. [**4 points**] Set

$$\mathbf{a}_1 = \begin{bmatrix}1\\0\\1\\3\end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix}-1\\1\\-1\\-2\end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix}1\\-2\\-3\\1\end{bmatrix}, \ \text{and } \mathbf{b} = \begin{bmatrix}-3\\0\\-3\\-7\end{bmatrix}.$$

Does $\mathbf{b}$ belong to $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?

Solution: We should reduce the following matrix to an E.F.:

$$\begin{bmatrix}1 & -1 & 1 & -3\\0 & 1 & -2 & 0\\1 & -1 & -3 & -3\\3 & -2 & 1 & -7\end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_1]{R_3 \to R_3 - R_1} \begin{bmatrix}1 & -1 & 1 & -3\\0 & 1 & -2 & 0\\0 & 0 & -4 & 0\\0 & 1 & -2 & 2\end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix}1 & -1 & 1 & -3\\0 & 1 & -2 & 0\\0 & 0 & -4 & 0\\0 & 0 & 0 & 2\end{bmatrix} \text{ E.F.}$$

Since the rightmost column of the E.F. is a pivot column, it follows that $\mathbf{b}$ is <u>not</u> a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

6. [**4 points**] For each of the following statements, indicate if it is true or false. You will receive 1 point for every correct answer and lose .5 points for every incorrect answer (but you cannot receive a negative mark on this question).

_____ Vectors of the form $\begin{bmatrix} 1 & -3 & 5 \\ 2 & -\frac{1}{2} & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is any vector in $\mathbb{R}^3$, are always in Span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, where $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ -\frac{1}{2} \\ 2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$.

Solution: True.

_____ Two row equivalent matrices always have the same number of pivot columns.

Solution: True.

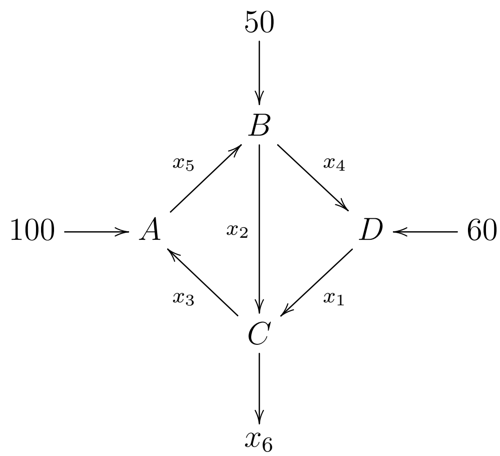
_____ A homogeneous linear system always has infinitely many solutions.

Solution: False.

_____ A linear system with 6 equations and 7 variables is always consistent.

Solution: False.

7. [**2 points**] Consider the traffic flow described by the following diagram. The letters A through D label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per hour.



Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 6$. (You do not need to solve the linear system.)

Solution:

$$\begin{cases} A: & 100 + x_3 = x_5 \\ B: & 50 + x_5 = x_2 + x_4 \\ C: & x_1 + x_2 = x_3 + x_6 \\ D: & x_4 + 60 = x_1 \\ \text{total}: & 100 + 50 + 60 = x_6 \end{cases}$$