University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II
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Final Exam – Solutions
April 2018

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of two parts. Questions 1 through 13 are *answer only*. For these questions, only your final answer will be considered for marks. Questions 14 through 20 are *long answer*. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) The number of points available for each question is indicated in square brackets.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) Write your student number at the top of each page in the space provided.

(f) You may use the second-to-last page of the exam as extra space for solutions. If you do so, indicate clearly on the page of the relevant question that you have continued your solution on the extra page.

(g) You may use the last page of the exam as scrap paper.

(h) Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you are complying with statement (h) above:

Signature: ______________________________

Good luck!

Please do not write in the table below.

Question	1–13	14	15	16	17	18	19	20	Total
Maximum	23	4	4	8	6	4	4	5	58
Grade									

Part A: Answer Only Questions

For Questions 1–13, only your final answer will be considered for marks. Write your final answers in the spaces provided.

1. [**2 points**] Consider the matrices

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & -\frac{1}{3} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4 \\ 3 & -3 \end{bmatrix}$$

Compute $B^T A$.

Answer: $B^T A = \begin{bmatrix} 7 & -10 & 2 \\ -10 & 13 & -11 \end{bmatrix}$

2. [**2 points**] Suppose that A is a 4×7 matrix and B is a 6×4 matrix. For each statement below, write '**T**' if the statement is true, and write '**F**' if the statement is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

T The columns of B can never form a basis of $\mathbb{R}^6$.

F $\text{Nul}(BA)$ is a subspace of $\mathbb{R}^6$.

T The reduced echelon form of A always has at least one non-pivot column.

T $\text{rank}(BA) = 7 - \dim \text{Nul}\,(BA)$.

3. **[2 points]** Suppose that A, B, and C are 4×4 matrices such that $\det(A) = -\frac{1}{2}$, $\det(B) = 5$, and $\det(-A^3BC^T) = 15$. Calculate $\det(C)$.

Answer: $\det(C) = -24$.

4. **[1 point]** Determine all values of $t \in \mathbb{R}$ such that the columns of the matrix

$$\begin{bmatrix} 3 & -4 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 5 & t & -1 \\ 0 & -5 & 3 & \frac{1}{2} \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

are linearly dependent.

Answer: $t = -3$

5. **[2 points]** Let $z = 1 - 2i$ and $w = i + 2$. Write the complex number $\frac{w}{1 + \overline{z}}$ in the form $a + bi$ where a and b are real numbers.

Answer: $\frac{3}{4} - \frac{1}{4}i$

6. [**2 points**] Let $A = \begin{bmatrix} -3 & 0 & 0 \\ -2 & 1 & 0 \\ 3-2i & -1+i & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Write down the eigenvalues of $(A+I)^2$ and their multiplicities.

Answer: 4 with multiplicity 2; 1 with multiplicity 1.

7. [**1 point**] Are the vectors

$$\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

linearly independent? Write 'Yes' if these vectors are linearly independent, and write 'No' if they are linearly dependent.

Answer: No, they are linearly dependent.

8. [**2 points**] Suppose we are given an $n \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$. Assume that the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent. For each statement below, write 'T' if it is true, and write 'F' if it is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

T Col A is not equal to $\mathbb{R}^n$.

F A is invertible.

T A has an eigenvector with eigenvalue $\lambda = 0$.

F $\mathbf{b} = \mathbf{0}_{n\times 1}$, where $\mathbf{0}_{n\times 1}$ denotes the zero vector in $\mathbb{R}^n$.

9. **[2 points]** Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in $\mathbb{R}^4$ such that $2\mathbf{a} - 3\mathbf{b} = \mathbf{b} + 5\mathbf{c}$. Justify that $\mathbf{a}$ belongs to $\text{Span}\{\mathbf{b}, \mathbf{c}\}$ by expressing $\mathbf{a}$ explicitly as a linear combination of $\mathbf{b}$ and $\mathbf{c}$.

Solution: $\mathbf{a} = 2\mathbf{b} + \frac{5}{2}\mathbf{c}$.

10. **[1 point]** Determine the value of the parameter t such that $\underline{\lambda = i}$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & t \\ -2 & 2i+1 \end{bmatrix}.$$

Answer: From $\det(A - iI) = 0$ we obtain the equation $2t + 2 = 0$ i.e., $t = -1$.

11. **[2 points]** For each of the following subsets of $\mathbb{R}^3$, write 'Y' if the set is a *subspace* of $\mathbb{R}^3$, and write 'N' if it is not. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

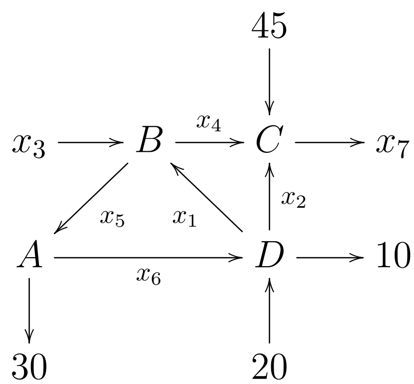
N $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1 + x_2 + 1 = 0 \right\}$

Y $\left\{ x \begin{bmatrix} -3 \\ 1 \\ -\frac{1}{2} \end{bmatrix} + y \begin{bmatrix} -2 \\ 0 \\ -7 \end{bmatrix} + z \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$

Y The eigenspace of the matrix $A = \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ -3 & 3 & 0 \\ 2 & 0 & 5 \end{bmatrix}$ corresponding to its eigenvalue $\lambda = 6$.

N $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \text{ and } y = x^2 \right\}$

12. [**2 points**] Consider the traffic flow described by the following diagram. The letters A through D label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.



Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 7$. (Do not solve the linear system.)

Solution:

$$\begin{cases} A: & x_5 = x_6 + 30 \\ B: & x_1 + x_3 = x_4 + x_5 \\ C: & x_2 + x_4 + 45 = x_7 \\ D: & x_6 + 20 = x_1 + x_2 + 10 \\ \text{Total}: & x_3 + 45 + 20 = x_7 + 30 + 10 \end{cases}$$

13. [**2 points**] Let A, B, and C be $n \times n$ invertible matrices. Solve the matrix equation $AX^TC - B = -BC$ for the matrix X.

Answer: $X = (C^{-1})^T B^T (A^{-1})^T - B^T (A^{-1})^T$.

Part B: Long Answer Questions

For Questions 14–20, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

14. **[4 points]** Is the following linear system consistent or inconsistent? If it is consistent, then write down the general solution in vector parametric form.

$$\begin{cases} x_1 & & -2x_3 & -x_4 & -2x_5 & = & 1 \\ & x_2 & +5x_3 & & +x_5 & = & -7 \\ 2x_1 & & -4x_3 & +x_4 & -7x_5 & = & 11 \end{cases}$$

Solution: We reduce the augmented matrix of the system to the REF:

$$\begin{bmatrix} 1 & 0 & -2 & -1 & -2 & | & 1 \\ 0 & 1 & 5 & 0 & 1 & | & -7 \\ 2 & 0 & -4 & 1 & -7 & | & 11 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 0 & -2 & -1 & -2 & | & 1 \\ 0 & 1 & 5 & 0 & 1 & | & -7 \\ 0 & 0 & 0 & 3 & -3 & | & 9 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & -2 & -1 & -2 & | & 1 \\ 0 & 1 & 5 & 0 & 1 & | & -7 \\ 0 & 0 & 0 & 1 & -1 & | & 3 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_3} \begin{bmatrix} 1 & 0 & -2 & 0 & -3 & | & 4 \\ 0 & 1 & 5 & 0 & 1 & | & -7 \\ 0 & 0 & 0 & 1 & -1 & | & 3 \end{bmatrix}$$

It follows that the system is consistent (since the rightmost column is not pivot). Moreover, x_1, x_2, x_4 are basic, x_3, x_5 are free. The general solution is

$$\begin{cases} x_1 = 2x_3 + 3x_5 + 4 \\ x_2 = -5x_3 - x_5 - 7 \\ x_3 = \text{free} \\ x_4 = x_5 + 3 \\ x_5 = \text{free} \end{cases}$$

The vector parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 + 3x_5 + 4 \\ -5x_3 - x_5 - 7 \\ x_3 \\ x_5 + 3 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -7 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

15. **[4 points]** Calculate the determinant of the following matrix using the method of co-factor expansion.

$$M = \begin{bmatrix} 0 & 15 & 0 & -1 \\ 5 & -3 & 7 & -2 \\ 0 & 3 & 0 & 0 \\ 1 & 11 & -1 & 9 \end{bmatrix}.$$

Solution: Expanding with respect to the 3rd row, and then with respect to the 1st row of the 3×3 matrix, we obtain

$$\begin{vmatrix} 0 & 15 & 0 & -1 \\ 5 & -3 & 7 & -2 \\ 0 & 3 & 0 & 0 \\ 1 & 11 & -1 & 9 \end{vmatrix} = -(3) \begin{vmatrix} 0 & 0 & -1 \\ 5 & 7 & -2 \\ 1 & -1 & 9 \end{vmatrix} = -(3)(-1) \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} = (3)(-12) = -36.$$

16. (a) [**4 points**] Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & -2 & 0 \\ -4 & 0 & 3 \end{bmatrix}$. Calculate the eigenvalues of A.

Solution: Starting with an expansion with respect to the second column, we have

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ -1 & -2-\lambda & 0 \\ -4 & 0 & 3-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -4 & 3-\lambda \end{vmatrix} = (-2-\lambda)(\lambda^2-4\lambda-5) = -(\lambda+2)(\lambda-5)(\lambda+1).$$

In fact the roots of $\lambda^2 - 4\lambda - 5$ are

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(-5)}}{2} = \begin{cases} 5 \\ -1 \end{cases}$$

Thus, the eigenvalues of A are -2 and -1 and 5.

(b) [**1 point**] Is A diagonalizable? You should justify your answer.

Solution: Yes because it has 3 distinct eigenvalues.

(c) [**3 points**] Let $B = \begin{bmatrix} 3 & 1 & 3 \\ -1 & 1 & -3 \\ 2 & 2 & 8 \end{bmatrix}$. Find a basis for the eigenspace of B corresponding to the eigenvalue $\lambda = 2$.

Solution: For the eigenvalue $\lambda = 2$, we have:

$$\left[\begin{array}{c|c} B-2I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ -1 & -1 & -3 & 0 \\ 2 & 2 & 6 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

and a basis of this eigenspace is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

17. The city of Quicheton has two culinary arts institutes: *ChefAcademie* and *RoyalToque*. In the beginning of January, the two institutes had equal student enrolment levels. During January, *ChefAcademie* updated its classes. As a result, at the end of January $\frac{1}{10}$ of *ChefAcademie*'s students switched to *RoyalToque*, while $\frac{1}{4}$ of *RoyalToque*'s students switched to *ChefAcademie*.

(a) **[1 point]** Write down the migration matrix M and the initial state vector $\mathbf{x}_0$ for this problem.

Solution:

$$M = \begin{bmatrix} .9 & .25 \\ .1 & .75 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Another possible migration matrix (if one reverses the order of the companies) is

$$M = \begin{bmatrix} .75 & .1 \\ .25 & .9 \end{bmatrix} \quad \vec{x}_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

(b) **[1 point]** What fraction of the total number of students will be enrolled at each of the institutes in the beginning of February?

Solution: $\vec{x}_1 = M\vec{x}_0 = \begin{bmatrix} \frac{23}{40} \\ \frac{17}{40} \end{bmatrix}$

(c) **[4 points]** If the same migration trend continues for several months, then in the long run what are the predicted enrolment ratios of each of the institutes? In your final answer, you should clearly indicate the fraction of students for each institute.

Solution: To find the steady-state vector, we must find an eigenvector of eigenvalue 1. We row reduce:

$$\left[\begin{array}{c|c} M - I & 0 \end{array}\right] = \left[\begin{array}{cc|c} -\frac{1}{10} & \frac{1}{4} & 0 \\ \frac{1}{10} & -\frac{1}{4} & 0 \end{array}\right] \xrightarrow[R_2 \to 10R_2]{R_1 \to -10R_1} \left[\begin{array}{cc|c} 1 & -\frac{5}{2} & 0 \\ 1 & -\frac{5}{2} & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution is thus

$$\vec{x} = x_2 \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}.$$

We choose the value of the free variable so that the sum of the entries of $\vec{x}$ is equal to one:

$$\left(\frac{5}{2} + 1\right) x_2 = 1 \implies x_2 = \frac{2}{7}.$$

Thus the steady-state vector is

$$\vec{x} = \begin{bmatrix} \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}.$$

Since M is regular stochastic, the long term behaviour is given by the steady-state vector. Thus, in the long term, $\frac{5}{7}$ of the total number of students will be enrolled in *ChefAcademie* and $\frac{2}{7}$ of students will be enrolled in *RoyalToque*.

18. **[4 points]** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 2 & 1 & -5 & 7 & -5 \\ -1 & 2 & -10 & 14 & 4 \end{bmatrix}.$$

Find a basis for Nul A.

Solution: We row reduce to find an echelon form of A.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 2 & 1 & -5 & 7 & -5 \\ -1 & 2 & -10 & 14 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & -5 & 7 & 1 \\ 0 & 2 & -10 & 14 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & -5 & 7 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & -5 & 7 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 + 3R_3]{R_2 \to R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Denoting the variables $x_1, \dots, x_5$ as usual, we realize that x_1, x_2, x_5 are basic and x_3, x_4 are free. It follows that the general solution is

$$\begin{cases} x_1 = 0 \\ x_2 = 5x_3 - 7x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \\ x_5 = 0 \end{cases}$$

and therefore the vector parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -7 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and the basis that we obtain for Nul A is

$$\left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

19. **[4 points]** Let

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & -1 & -3 \\ 2 & 0 & -6 \end{bmatrix}$$

Find the inverse of A.

Solution: We row reduce the super-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 2 & 0 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 4 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 & -4 & 1 \end{array}\right] \xrightarrow{R_3 \to (-\frac{1}{2})R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + 4R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -3 & 8 & -2 \\ 0 & 1 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 2 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 6 & -1 \\ 0 & 1 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 2 & -\frac{1}{2} \end{array}\right]$$

Thus A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -3 & 6 & -1 \\ 0 & -1 & \frac{1}{2} \\ -1 & 2 & -\frac{1}{2} \end{bmatrix}.$$

20. An economy consists of two sectors: Agriculture and Service. In order to produce one unit, the Agriculture sector consumes $\frac{1}{3}$ units from Agriculture and $\frac{1}{3}$ units from Service. Also, in order to produce one unit, the Service sector consumes $\frac{2}{3}$ units from Agriculture and $\frac{1}{6}$ units from Service.

(a) **[1 point]** Write the consumption matrix for this economy.

Solution:

$$C = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

(b) **[1 point]** Write the Leontief Input-Output Model production equation.

Solution: The equation is $\vec{x} = C\vec{x} + \vec{d}$, or $(I - C)\vec{x} = \vec{d}$.

(c) **[3 points]** Determine the production levels needed to satisfy a final demand of 12 units from the Agriculture sector and 6 units from the Service sector.

Solution: We solve the equation $A\vec{x} = \vec{d}$, where

$$A = I_2 - C = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

We have

$$\det A = \frac{10}{18} - \frac{2}{9} = \frac{1}{3} \neq 0.$$

Thus A is invertible and

$$A^{-1} = 3 \begin{bmatrix} \frac{5}{6} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 2 \\ 1 & 2 \end{bmatrix}.$$

Therefore,

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} \frac{5}{2} & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 42 \\ 24 \end{bmatrix}.$$

So the Agriculture sector needs to produce 42 units and the Service sector needs to produce 24 units.