University of Ottawa

Department of Mathematics and Statistics

MAT 1302B : Mathematical Methods II

Professor: Hadi Salmasian

Third Midterm Exam – Version A

March 24, 2017

Surname ______________________ First Name ______________________

Student # ______________________ DGD ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ______________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	2	3	5	3	5	21
Grade							

1. [**3 points**] For each of the following sets, write **Yes** if the set is a subspace of $\mathbb{R}^n$ for the *given* value of n, and write **No** if it is not. You will receive .5 points for each correct answer and lose .25 points for each incorrect answer.

No Nul A where A is a 4×5 matrix, $n = 4$.

Yes Span $\left\{ \begin{bmatrix} 2 \\ -1 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \frac{1}{3} \end{bmatrix} \right\}$, $n = 3$.

Yes $\left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} 2 & -1 & 3 \\ \frac{1}{3} & 2 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, $n = 3$.

No Col B where B is a 6×4 matrix, $n = 4$.

Yes $\left\{ \begin{bmatrix} t \\ -3t \\ 0 \\ 2t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $n = 4$.

No $\left\{ \begin{bmatrix} s \\ -s \\ 0 \\ -2s \end{bmatrix} \;\middle|\; s \geq 1 \right\}$, $n = 4$.

2. [**2 points**] For each of the following statements, indicate if it is true (**T**) or false (**F**). You will receive .5 points for each correct answer, and will lose .25 points for each incorrect answer.

F Every list of n distinct vectors in $\mathbb{R}^n$ forms a basis of $\mathbb{R}^n$.

T If two rows of an $n \times n$ matrix A are equal, then $\det(A) = 0$.

T If A is a 6×6 matrix, then $\det(-A) = \det(A^T)$.

F The rank of a matrix A is equal to the number of non-pivot columns of A.

3. [**3 points**] Calculate the following determinant. You should write your answer in the form $a + bi$ where $a, b \in \mathbb{R}$.

$$\begin{vmatrix} 1+i & 1 & -\dfrac{3}{5+i} \\ 0 & \dfrac{1}{2+i} & -2i+1 \\ 0 & 0 & -2 \end{vmatrix} =$$

Solution: The matrix is triangular, so its determinant is the product of diagonal entries, that is,

$$\frac{-2(1+i)}{2+i} = \frac{-2(1+i)(2-i)}{(2+i)(2-i)} = \frac{-6-2i}{5} = -\frac{6}{5} - \frac{2}{5}i$$

4. [**5 points**] Calculate the determinant of the matrix A given below using the row reduction method.

$$A = \begin{bmatrix} 2 & -1 & 2 & 0 & -3 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & -1 & 1 & 3 & -6 \\ 0 & 0 & 3 & 6 & 3 \\ -2 & 1 & 1 & -4 & 3 \end{bmatrix}$$

Solution: The best method for such a large matrix is the row reduction method:

$$\begin{bmatrix} 2 & -1 & 2 & 0 & -3 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & -1 & 1 & 3 & -6 \\ 0 & 0 & 3 & 6 & 3 \\ -2 & 1 & 1 & -4 & 3 \end{bmatrix} \xrightarrow{R_5 \to R_5 + R_1} \begin{bmatrix} 2 & -1 & 2 & 0 & -3 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & -1 & 1 & 3 & -6 \\ 0 & 0 & 3 & 6 & 3 \\ 0 & 0 & 3 & -4 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightsquigarrow R_3} \begin{bmatrix} 2 & -1 & 2 & 0 & -3 \\ 0 & -1 & 1 & 3 & -6 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 6 & 3 \\ 0 & 0 & 3 & -4 & 0 \end{bmatrix} \xrightarrow[R_5 \to R_5 - R_3]{R_4 \to R_4 - R_3} \begin{bmatrix} 2 & -1 & 2 & 0 & -3 \\ 0 & -1 & 1 & 3 & -6 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & -5 & -2 \end{bmatrix}$$

$$\xrightarrow{R_5 \to R_5 + R_4} \begin{bmatrix} 2 & -1 & 2 & 0 & -3 \\ 0 & -1 & 1 & 3 & -6 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The last matrix is triangular and therefore its determinant is the product of diagonal entries, i.e., 30. Since we used one interchange throughout, the determinant of A is equal to $(-1)(30) = -30$.

5. [**3 points**] Let

$$A = \begin{bmatrix} 2 & 5 & 1 & 3 & -1 \\ -2 & -5 & -1 & -6 & 3 \\ 4 & 10 & 2 & 6 & -2 \end{bmatrix}.$$

Find a basis for Col A.

Solution: We compute an Echelon Form of A:

$$\begin{bmatrix} 2 & 5 & 1 & 3 & -1 \\ -2 & -5 & -1 & -6 & 3 \\ 4 & 10 & 2 & 6 & -2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 2 & 5 & 1 & 3 & -1 \\ 0 & 0 & 0 & -3 & 2 \\ 4 & 10 & 2 & 6 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 2 & 5 & 1 & 3 & -1 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This the 1st and 4th columns of A are pivot columns. It follows that a basis for Col A is given by

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix} \right\}$$

6. Suppose that

$$A = \begin{bmatrix} -2 & 4 & 1 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \\ 2 & -4 & -1 & -1 & -13 \end{bmatrix}$$

(a) [**4 points**] Find a basis for Nul A.

Solution:

$$\begin{bmatrix} -2 & 4 & 1 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \\ 2 & -4 & -1 & -1 & -13 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} -2 & 4 & 1 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & -1 & -5 \end{bmatrix} \xrightarrow{R_3 \to -R_3} \begin{bmatrix} -2 & 4 & 1 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} -2 & 4 & 1 & 0 & 8 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} -2 & 4 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_1 \to -\frac{1}{2} R_1} \begin{bmatrix} 1 & -2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \text{ R.E.F.}$$

Therefore if we denote the variables by $x_1, \dots, x_5$ as usual, then x_1, x_3, x_4 are basic, whereas x_2, x_5 are free. are the general solution is

$$\begin{cases} x_1 = 2x_2 + 3x_5 \\ x_2 = \text{free} \\ x_3 = -2x_5 \\ x_4 = -5x_5 \\ x_5 = \text{free} \end{cases}$$

The vector parametric form of the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + 3x_5 \\ x_2 \\ -2x_5 \\ -5x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -2 \\ -5 \\ 1 \end{bmatrix}$$

Therefore the basis for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ -5 \\ 1 \end{bmatrix} \right\}$$

(b) [**1 point**] Determine rank (A). Justify your answer.

Solution: By the rank theorem, rank $A = 5 - \dim \operatorname{Nul} A = 3$.