University of Ottawa

Department of Mathematics and Statistics

MAT 1302B : Mathematical Methods II

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Second Midterm Exam – Version A

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Surname ______________________ First Name ______________________

Student # ______________________ DGD ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ______________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	5	4	5	5	3	26
Grade							

1. Let

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ -3 & \frac{1}{2} \end{bmatrix}$$

Also, let I_2 denote the 2×2 identity matrix. In parts (a) and (b) below, calculate the given matrix expressions explicitly.

(a) **[2 points]** $A^T B - 2I_2$

Solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ -3 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & \frac{1}{2} \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ 2 & -5 \end{bmatrix}$$

(b) **[2 points]** $(A^T A)^{-1}$

Solution:

$$\left(\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$$

2. [**5 points**] For each of the following statements, indicate if it is true (**T**) or false (**F**). You will receive 1 point for each correct answer and $-.5$ points for each incorrect answer (but you cannot receive a negative score on this question).

T The reduced echelon form of an invertible $n \times n$ matrix is the identity matrix I_n.

F If A is an $n \times n$ matrix such that the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then A is invertible.

F The vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right\}$ are linearly independent.

T For every two $n \times n$ matrices A and B, we have $(A+B)^T = B^T + A^T$.

F For three $n \times n$ matrices A, B, and C, if $AB = AC$ then we necessarily have $B = C$.

3. [**4 points**] Is the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

invertible? If the answer is yes, determine its inverse.

Solution: We row reduce the super-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

We conclude that the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

4. An economy consists of two sectors: Agriculture and Manufacturing. In order to produce 1 unit of output, the Agriculture sector uses $\frac{5}{8}$ units from the Agriculture and $\frac{1}{8}$ units from the Manufacturing sector. Further, to produce 1 unit of output, the Manufacturing sector uses $\frac{1}{4}$ units from the Agriculture and $\frac{1}{4}$ units from the Manufacturing sector.

(a) [**1 point**] Write down the consumption matrix C for this economy.

Solution:

$$C = \begin{bmatrix} \frac{5}{8} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

(b) [**1 point**] What intermediate demands are created if Manufacturing is supposed to produce 16 units, and simultaneously Agriculture is supposed to produce 8 units?

The intermediate demands are _______ units from Agriculture and _______ units from Manufacturing.

Solution:

$$8 \begin{bmatrix} \frac{5}{8} \\ \frac{1}{8} \end{bmatrix} + 16 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

The intermediate demands are 9 units from Agriculture and 5 units from Manufacturing.

(c) [**3 points**] Determine the production levels required to meet a final demand of 4 units from the Agriculture sector and 6 units from the Manufacturing sector.

The required production levels are _______ units from Agriculture and _______ units from Manufacturing.

Solution: We should solve the equation $A\mathbf{x} = \mathbf{d}$ where

$$A = I_2 - C = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Now we have $\det A = \frac{9}{32} - \frac{1}{32} = \frac{8}{32} = \frac{1}{4} \neq 0$, thus A is invertible and

$$A^{-1} = 4 \times \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

It follows that, $\mathbf{x} = A^{-1}\mathbf{d} = \begin{bmatrix} 3 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$

The production levels are 18 units from Agriculture and 11 units from Manufacturing.

5. [**5 points**] Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\3\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\2\\4\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}-2\\-1\\-5\end{bmatrix}$$

Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly independent? If they are linearly dependent, then write down a linear dependence relation for these vectors. You should justify your answers.

Solution: We form the matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ and calculate the reduced echelon form:

$$A = \begin{bmatrix}1 & 2 & -2\\0 & 2 & -1\\3 & 4 & -5\end{bmatrix} \xrightarrow{R_3 \to R_2 - 3R_1} \begin{bmatrix}1 & 2 & -2\\0 & 2 & -1\\0 & -2 & 1\end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix}1 & 2 & -2\\0 & 2 & -1\\0 & 0 & 0\end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix}1 & 2 & -2\\0 & 1 & -\frac{1}{2}\\0 & 0 & 0\end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix}1 & 0 & -1\\0 & 1 & -\frac{1}{2}\\0 & 0 & 0\end{bmatrix}$$

Since the third column is not a pivot column, the homogeneous linear system with coefficient matrix A has infinitely many solutions. Therefore the given vectors are linearly dependent.

To find the linear dependence relation, we should solve the homogeneous system with coefficient matrix A. The linear system corresponding to the RREF of A is:

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \\ x_3 = \text{free} \end{cases}$$

Therefore the general solution of the system is given by:

$$\begin{cases} x_1 = x_3 \\ x_2 = \frac{1}{2}x_3 \\ x_3 = \text{free} \end{cases}$$

Now we can choose the value of x_3; e.g., $x_3 = 1$, and then we find a particular solution $x_1 = 1,\ x_2 = \frac{1}{2},\ x_3 = 1$, which results in the linear dependence relation $\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$.

6. [**3 points**] Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, and let B be a 2×3 matrix. Suppose that the second row of $(AB)^T$ is

$$\mathbf{v} = \begin{bmatrix} -1 & 3 \end{bmatrix}.$$

Compute the second column of B.

Solution: We have

$$(AB)^T = \begin{bmatrix} * & * \\ -1 & 3 \\ * & * \end{bmatrix}$$

and therefore if we write B in column form as $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ then

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} * & -1 & * \\ * & 3 & * \end{bmatrix}.$$

It follows that

$$A\mathbf{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Therefore

$$\mathbf{b}_2 = A^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$