University of Ottawa
Department of Mathematics and Statistics

MAT 1302B : Mathematical Methods II
Professor: Hadi Salmasian

First Midterm Exam – Version A

February 3, 2017

Surname ________________________ First Name ______________________

Student # ______________________ DGD ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ________________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	3	2	4	5	3	21
Grade							

1. [**4 points**] Determine if the matrix equation

$$\begin{bmatrix} 1 & -2 & 2 & -1 \\ -1 & 2 & 0 & -1 \\ 2 & -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -4 \end{bmatrix}$$

is consistent or inconsistent. If the linear system is consistent, write its general solution in vector parametric form.

Solution:
Reduced echelon form of the augmented matrix of the system is obtained as follows:

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -2 \\ -1 & 2 & 0 & -1 & -4 \\ 2 & -4 & 4 & -2 & -4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -2 \\ 0 & 0 & 2 & -2 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ E.F.}$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ R.E.F.}$$

From the E.F. or the R.E.F. it is clear that the system is consistent. The pivot columns are the first and the third columns, so x and z are basic, and y and t are free. The R.E.F. is the augmented matrix of the linear system

$$\begin{array}{ccccc} x & -2y & & +t & = & 4 \\ & & z & -t & = & -3 \end{array}$$

Therefore the general solution is

$$\begin{cases} x = 2y - t + 4 \\ y : \text{free} \\ z = t - 3 \\ t : \text{free} \end{cases}$$

and in vector parametric form:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y - t + 4 \\ y \\ t - 3 \\ t \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

2. [**3 points**] Determine all values of the parameter h such that the linear system

$$\begin{cases} 2x_1 &= 2x_2 + x_3 + 4 \\ x_1 &= x_2 - hx_3 - 3 \end{cases}$$

is inconsistent. You should justify your answer.

Solution: The linear system can be written in the standard form as

$$\begin{cases} 2x_1 - 2x_2 - x_3 &= 4 \\ x_1 - x_2 + hx_3 &= -3 \end{cases}$$

The augmented matrix of this system can be reduced to an E.F. as follows

$$\begin{bmatrix} 2 & -2 & -1 & | & 4 \\ 1 & -1 & h & | & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & -2 & -1 & | & 4 \\ 0 & 0 & h + \frac{1}{2} & | & -5 \end{bmatrix}$$

The system is inconsistent if and only if $h = -\frac{1}{2}$.

3. Compute the following:

(a) [**1 point**] $\begin{bmatrix} 4 & -3 & -1 \\ 2 & 0 & -1 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Solution: $\begin{bmatrix} -12 \\ -5 \\ 12 \end{bmatrix}$

(b) [**1 point**] $-3\mathbf{u} + 2\mathbf{v}$ where $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

Solution: $\begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$

4. [**4 points**] Set

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \ \text{and } \mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$

Does $\mathbf{b}$ belong to $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? You should justify your answer.

Solution: We should reduce the following matrix to an E.F.:

$$\begin{bmatrix} 1 & 2 & -1 & -3 \\ -2 & 1 & 0 & 1 \\ 3 & 1 & -1 & -4 \\ -1 & 3 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 + R_1]{\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1}} \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & -5 & 2 & 5 \\ 0 & 5 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{\substack{R_3 \to R_3 + R_2 \\ R_4 \to R_4 - R_2}} \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightsquigarrow R_4} \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column is not a pivot column, it follows that $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

5. [**5 points**] For each of the following statements, indicate if it is true or false. You will receive 1 point for every correct answer and lose .5 points for every incorrect answer (but you cannot receive a negative mark on this question).

_____ A homogeneous linear system with n equations and n variables always has a unique solution.

Solution: False.

_____ For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$, the vector $-2\mathbf{a}+\mathbf{b}-3\mathbf{c}$ always belongs to Span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

Solution: True.

_____ A consistent linear system with a 4×6 augmented matrix will have infinitely many solutions.

Solution: True.

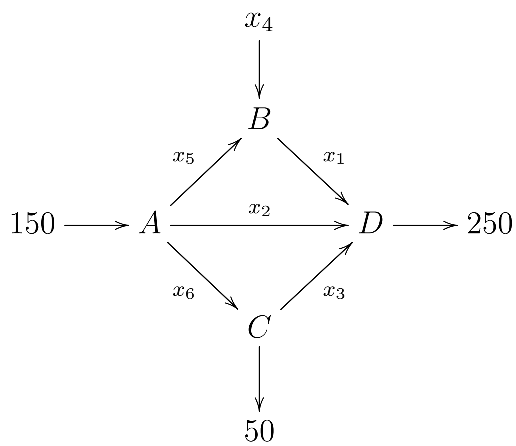
_____ If a 4×5 matrix A has 4 pivot positions, then for every vector $\mathbf{b}$ in $\mathbb{R}^4$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Solution: True.

_____ Two row equivalent matrices which are both in echelon form must be equal.

Solution: False.

6. Consider the traffic flow described by the following diagram. The letters A through D label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per hour.



(a) [**2 points**] Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 6$. (You do not need to solve the linear system.)

Solution:

$$\begin{cases} A: & 150 = x_2 + x_5 + x_6 \\ B: & x_4 + x_5 = x_1 \\ C: & x_6 = x_3 + 50 \\ D: & x_1 + x_2 + x_3 = 250 \\ \text{total}: & x_4 + 150 = 250 + 50 \end{cases}$$

(b) [**1 point**] The reduced row echelon form of the augmented matrix corresponding to the linear system in part (a) is:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 150 \\ 0 & 1 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 1 & 0 & 0 & -1 & -50 \\ 0 & 0 & 0 & 1 & 0 & 0 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Write down the general flow pattern.

Solution: The general flow pattern is the general solution of the linear system, given by

$$\begin{cases} x_1 &= x_5 + 150 \\ x_2 &= -x_5 - x_6 + 150 \\ x_3 &= x_6 - 50 \\ x_4 &= 150 \\ x_5 &: \text{free} \\ x_6 &: \text{free} \end{cases}$$