University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

Instructor: Hadi Salmasian

Final Exam

April 2017

Surname ____________________ First Name ____________________

Student # ____________________ Seat # ____________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of two parts. Questions 1 through 14 are *answer only*. For these questions, only your final answer will be considered for marks. Questions 15 through 21 are *long answer*. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) The number of points available for each question is indicated in square brackets.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) Write your student number at the top of each page in the space provided.

(f) You may use the second-to-last page of the exam as extra space for solutions. If you do so, indicate clearly on the page of the relevant question that you have continued your solution on the extra page.

(g) You may use the last page of the exam as scrap paper.

(h) Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with statement (h) above:

Signature: ____________________________

Good luck!

Please do not write in the table below.

Question	1–14	15	16	17	18	19	20	21	Total
Maximum	27	5	4	8	6	3	4	3	60
Grade									

Part A: Answer Only Questions

For Questions 1–14, only your final answer will be considered for marks. If applicable, write your final answers in the spaces provided.

1. **[2 points]** Consider the matrices

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Compute $-4I_2 + A^T B$.

Final Answer:____________

2. **[2 points]** Let $z = 2i + 3$ and $w = 1 - i$. Write the complex number $\frac{-2}{z - \overline{w}}$ in the form $a + bi$ where a and b are real numbers.

Final Answer:____________

3. **[2 points]** Let A, B, and C be 4×4 matrices such that $\det A = 2$, $\det B = -\frac{1}{3}$, and $\det(C) = 6$. Calculate $\det(-2AB^T CA^{-3})$.

Final Answer:____________

4. **[2 points]** Determine all values of $t \in \mathbb{R}$ such that the linear system

$$\begin{cases} 2y_1 - 3y_2 + \frac{1}{2}y_3 = -2 \\ 4y_1 - 6y_2 + 2ty_3 = -3 \end{cases}$$

is consistent.

Final Answer:____________

5. **[2 points]** Let A and B be two $m \times n$ matrices such that $n > m \geq 1$. Each of the statements below is either always true or always false. For each statement below, write '**T**' if the statement is always true, and write '**F**' if the statement is always false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

____ $\text{Col}(A^T)$ is a subspace of $\mathbb{R}^m$.

____ $\text{Nul}(B) \neq \{0\}$

____ $\det(AB^T) = \det(BA^T)$

____ $\text{rank}(AB^T) + \dim \text{Nul}\,(AB^T) = n$.

6. **[2 points]** Let $A = \begin{bmatrix} 1+i & 0 & 0 \\ -2 & 1 - \dfrac{1}{i} & 0 \\ 1+2i & -i & -1 \end{bmatrix}$. Write down the eigenvalues of A and their multiplicities. You should write the eigenvalues in the form $a + bi$, where $a, b \in \mathbb{R}$.

Final Answer:____________

7. **[1 point]** Suppose that A is a 6×6 matrix which has 4 non-pivot columns. Write down the dimension of the subspace $W = \left\{ \mathbf{v} \in \mathbb{R}^6 \ : \ \text{the equation } A\mathbf{x} = \mathbf{v} \text{ has a solution } \mathbf{x} \in \mathbb{R}^6 \right\}$.

Final Answer:____________

8. [**2 points**] For each of the following subsets of $\mathbb{R}^3$, write 'Y' if the set is a subspace of $\mathbb{R}^3$, and write 'N' if it is not. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

____ $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; -3x_1 + x_2 - 2x_3 = 0 \right\}$

____ $\left\{ \begin{bmatrix} x+y \\ x-y \\ 2x-y \end{bmatrix} \;\middle|\; x, y \geq 0 \right\}$

____ The line in $\mathbb{R}^3$ that passes through the points $(1,0,0)$ and $(0,0,0)$.

____ $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{bmatrix} -2 & 3 & -2 \\ 0 & -2 & 2 \\ -\frac{1}{2} & -\frac{1}{2} & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix} \right\}$

9. [**1 point**] Suppose that A and B are 4×7 matrices, and suppose that $C = A + \frac{1}{2}B$. Let $\mathbf{x}$ be a vector in $\mathbb{R}^7$ such that

$$A\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \end{bmatrix} \quad \text{and} \quad B\mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ -6 \\ 4 \end{bmatrix}.$$

Calculate $C\mathbf{x}$.

Final Answer:____________

10. [**3 points**] Let A be an $n \times n$ matrix, and consider the sentence

"A is invertible." $(\star)$

For each statement below, write 'Y' if it is equivalent to the sentence $(\star)$, and write 'N' if it is not equivalent to $(\star)$. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

____ $\det(A) \neq 0$.

____ The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is consistent.

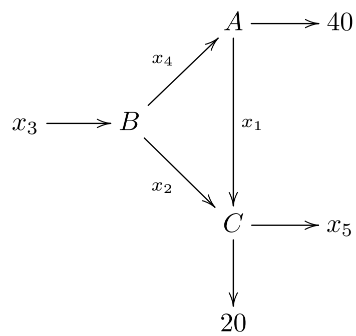
____ Every column vector in $\mathbb{R}^n$ can be expressed as a linear combination of the columns of A.

____ The columns of A form a basis for $\mathbb{R}^n$.

____ The reduced echelon form of A has non-pivot columns.

____ $\operatorname{rank} A = 0$.

11. **[2 points]** Consider the traffic flow described by the following diagram. The letters A through C label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.



Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 5$. (Do not solve the linear system.)

12. **[2 points]** Let A be an 8×8 matrix such that the characteristic equation of A is

$$\lambda^8 - 3\lambda^3 + 2\lambda^2 = 0.$$

Also recall that I_8 denotes the 8×8 identity matrix. For each statement below, write 'T' if the statement is true, and write 'F' if the statement is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

____ A is invertible.

____ There exists a nonzero vector $\mathbf{x} \in \mathbb{R}^8$ such that $A\mathbf{x} = \mathbf{x}$.

____ The eigenvalue $\lambda = 0$ of A has multiplicity 2.

____ The homogeneous linear system $(A - 2I_8)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

13. **[2 points]** Write down $\det(A^3)$, where

$$A = \begin{bmatrix} 1 & 2 & -\frac{3}{5} & 0 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Final Answer:____________

14. **[2 points]** Let A and B be $n \times n$ invertible matrices. Solve the matrix equation $AB^T XB^2 + BA^T = 0$ for the matrix X.

Final Answer:____________

Part B: Long Answer Questions

For Questions 15–21, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

15. (a) [**4 points**] Is the following linear system consistent or inconsistent? If it is consistent, then write down the general solution in vector parametric form.

$$\begin{cases} x_1 + 2x_3 + 9x_4 &= 4x_2 + 1 \\ x_3 + 5x_4 &= -1 \\ 8x_2 - 2x_3 &= 2x_1 + 8x_4 - 4 \end{cases}$$

(b) [**1 point**] Using your solution to part (a), write down the general solution of the following linear system in vector parametric form.

$$\begin{bmatrix} 1 & -4 & 2 & 9 \\ 0 & 0 & 1 & 5 \\ -2 & 8 & -2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hint: Compare the augemented matrices of the two systems.

16. [**4 points**] Calculate the determinant of the following matrix using the method of cofactor expansion.

$$M = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 0 & 0 & 1 \\ 7 & -2 & 11 & -6 \\ 1 & 0 & -3 & 1 \end{bmatrix}.$$

17. Consider the matrix $B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) **[3 points]** By finding the roots of the characteristic polynomial, show that the eigenvalues of B are -1 and 2.

(b) **[4 points]** For each of the eigenvalues of B found in part (a), find a basis of the corresponding eigenspace. (There is additional space for answering this part on the next page.)

There is extra space for this question on the next page.

(Extra space for part (b).)

(c) **[1 point]** Is B a diagonalizable matrix? Justify your answer.

18. A European country has two private jet rental companies: *AeroNocia* and *GloboWings*. In September, $\frac{1}{4}$ of the country's businessmen flew with AeroNocia, while $\frac{3}{4}$ of the businessmen flew with GloboWings. After three months of tough marketing campaign between the two companies, $\frac{1}{4}$ of AeroNocia's customers switched to GloboWings, while $\frac{1}{2}$ of GloboWings' customers switched to AeroNocia.

(a) **[1 point]** Write down the migration matrix M and the initial state vector $\vec{x}_0$ for this problem.

(b) **[1 point]** What is the market share of each of the companies at the end of the three-month marketing campaign?

(c) **[4 points]** Suppose that the same marketing campaign continues for several more three-month time periods, and its effect on customer migration remains the same. Then, in the long run, what is the predicted market share of each company? Put down your final answer in the following blank spots and write your detailed solution below.

In the long run, the market share of AeroNocia will be _______, and the market share of GloboWings will be _______.

19. [**3 points**] Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -4 & 2 & 3 \\ 1 & 1 & -2 & 0 & 8 \\ 2 & 0 & -8 & 5 & 5 \end{bmatrix}.$$

Find a basis for $\operatorname{Nul} A$.

20. [**4 points**] Let

$$A = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -3 & -3 \\ -1 & 6 & 4 \end{bmatrix}$$

Find the inverse of A.

21. **[3 points]** Determine the dimension of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}.$$

Extra page for answers.

This page is intentionally left blank. You may use it as scrap paper.